Lower bounds for tropical circuits and the power of pure dynamic programming

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UNIVERSITY OF COPENHAGEN

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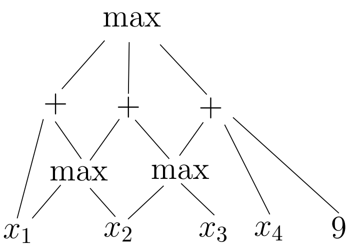
Tropical circuits



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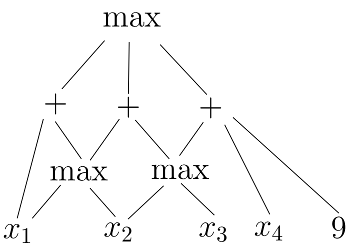

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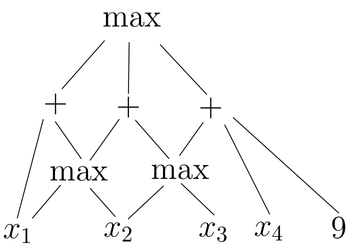

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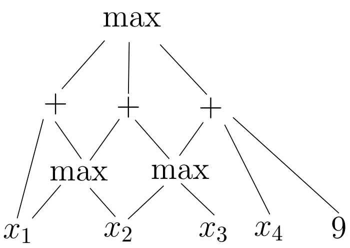

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- Tropical circuits: Circuits over the $(\max, +)$-semiring (or the $(\min, +)$-semiring)
- First studied by [Jerrum & Snir, '82], recently by [Jukna '16, Jukna '17, Jukna & Seiwert '19, Jukna & Seiwert '20, K. '21, Jukna '23, Kluk & Nederlof '25])

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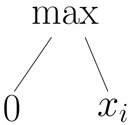

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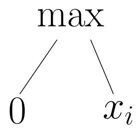
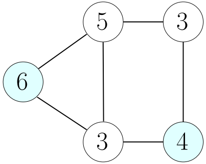
- Rest of this talk: Assume inputs can be negative (they are arbitrary reals)



Maximum weight independent set (MWIS)



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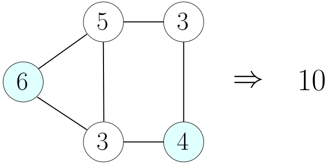

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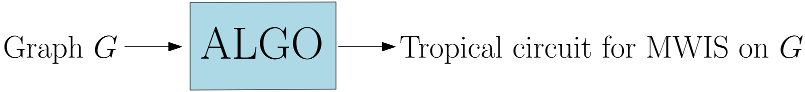

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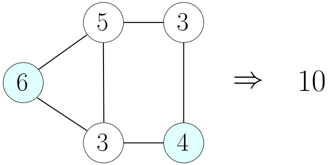

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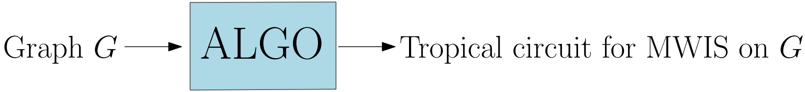

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- Bounded cliquewidth
- Chordal graphs
- P_5-free graphs
- Graphs with polynomial number of minimal separators

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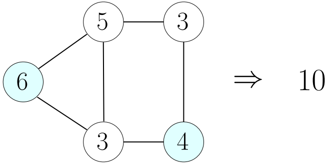

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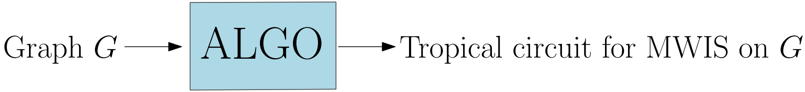

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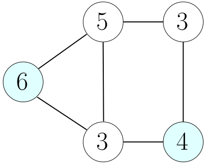

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Graph G → ALGO → Tropical circuit for MWIS on G

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Non-examples:

- Perfect graphs (incl. bipartite)
- Line graphs (and their generalizations)

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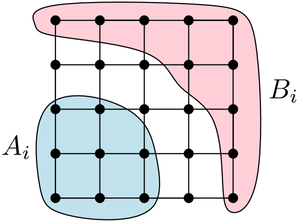

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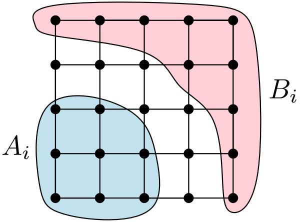

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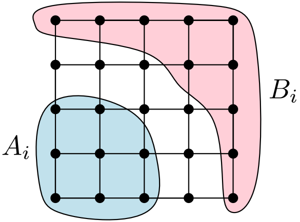

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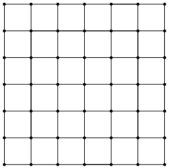

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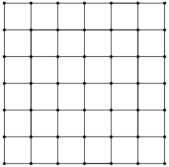

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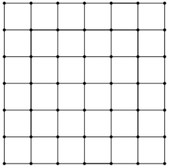

Family of related conjectures and lot of partial progress [K. '23], [Bonamy, Bonnet, Déprés, Esperet, Geniet, Hilaire, Thomassé, Wesolek '23], [K. & Lokshtanov '24], [Chudnovsky, Gartland, Hajebi, Lokshtanov, Spirkl '25]

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Open: Improve to $2^{\Omega(k \log k)}$

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- The power of idempotence
- Quasipolynomial separation should be doable, but exponential not clear

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1. Characterize which graphs have poly-size tropical circuits for MWIS
2. Prove $2^{\Omega(k \log k)}$ lower bound for TSP on treewidth-k graphs

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2. Prove $2^{\Omega(k \log k)}$ lower bound for TSP on treewidth-k graphs
3. Separation between tropical and monotone arithmetic circuits for multilinear polynomials with $0, 1$-coefficients

Conclusion

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Thank you!