Tutorial: New algorithms for computing treewidth

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STWOR

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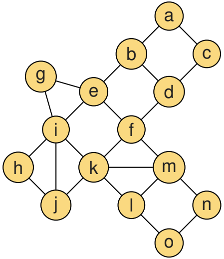
Treewidth

- Measures how close a graph is to a tree



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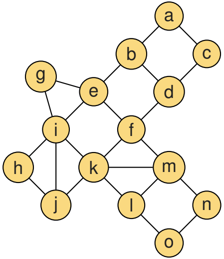

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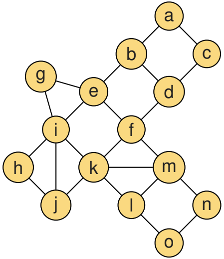

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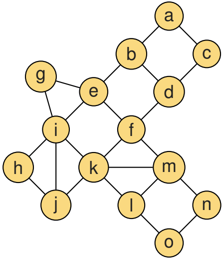

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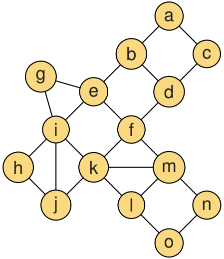

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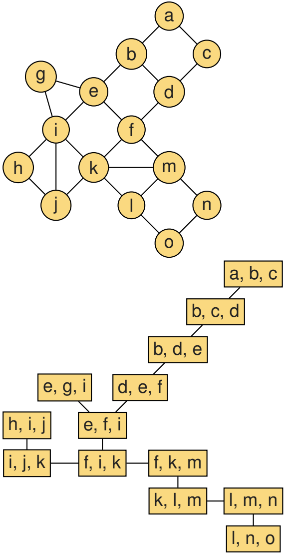

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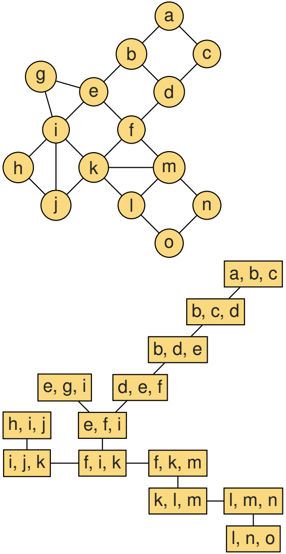

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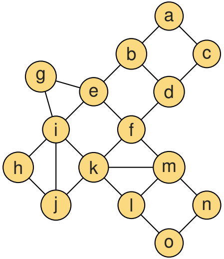

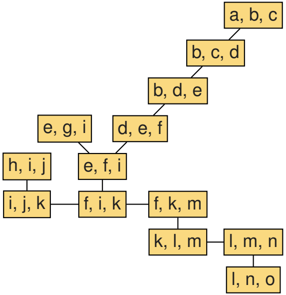

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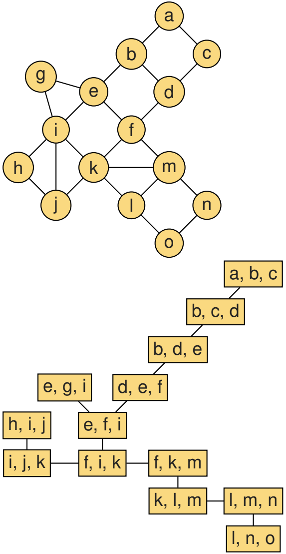

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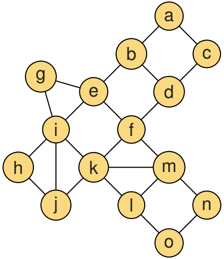

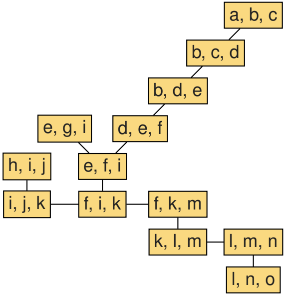

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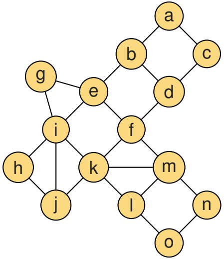

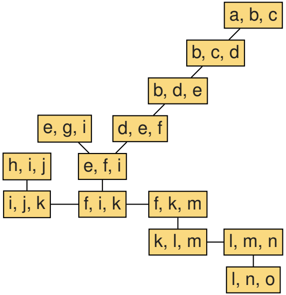

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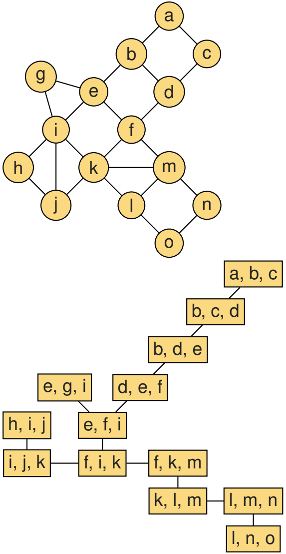
[Robertson & Seymour '84, Arnborg & Proskurowski '89, Bertele & Brioschi '72, Halin '76]



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Many more: [ACP'87,MT'91,Lagergren'96,Reed'92,Amir'10,FHL'08,FTV'15,FLS'18,BF'21,BF'22]

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(first non-trivial algorithm in this setting for $k \geq 3$)

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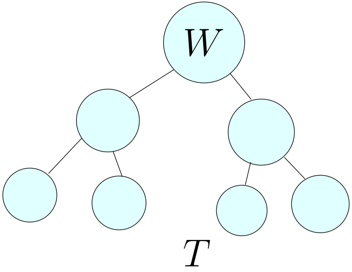
Setting

We have a tree decomposition T whose largest bag is W



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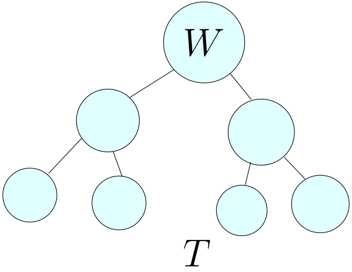

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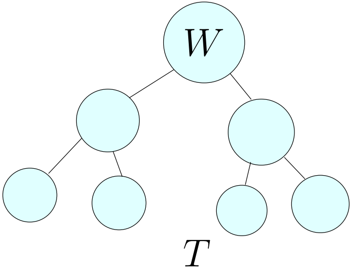

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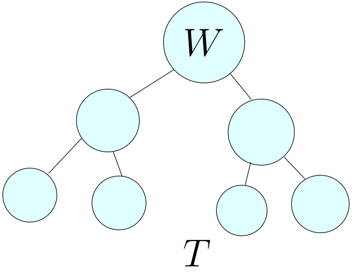

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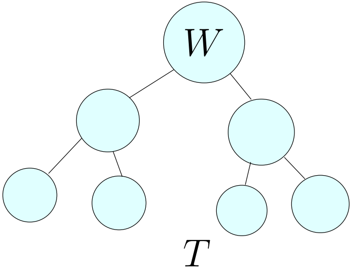

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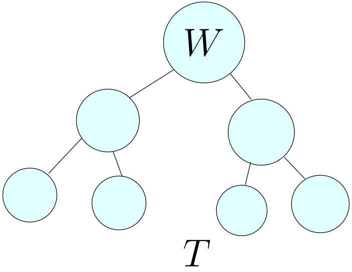
(by [Bodlaender'93] we can assume to start with a decomposition of width $\mathcal{O}(\text{tw}(G))$)



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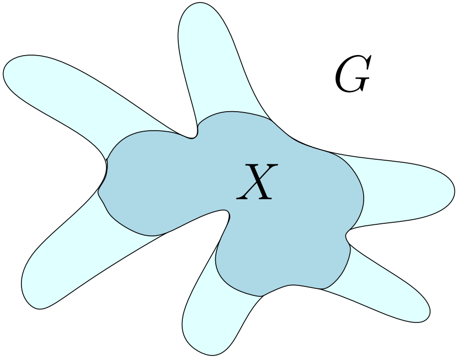

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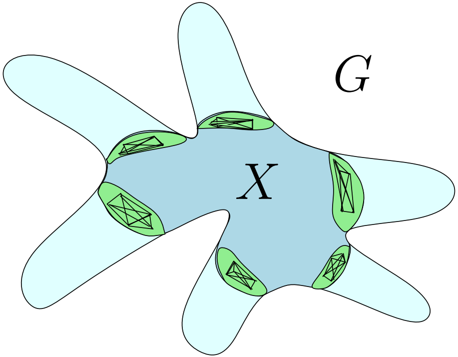

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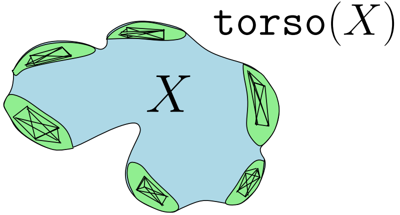

- Make neighborhoods of components of $G \setminus X$ into cliques
- Delete $V(G) \setminus X$

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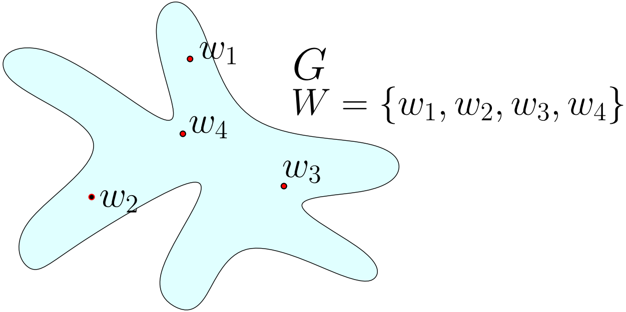

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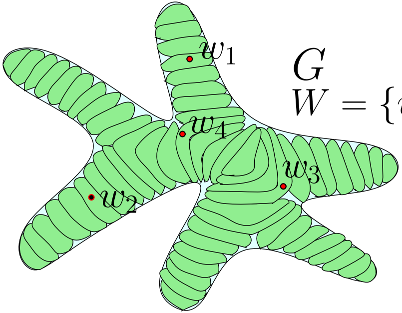

G

$W = \{w_1, w_2, w_3, w_4\}$

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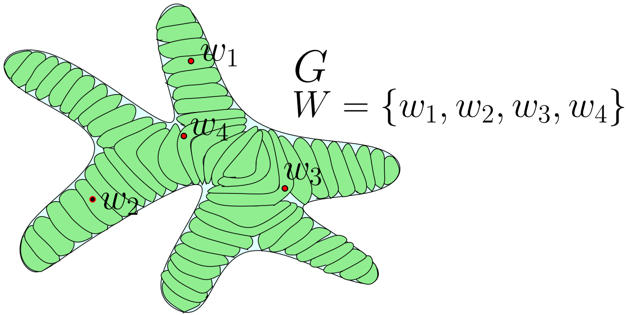

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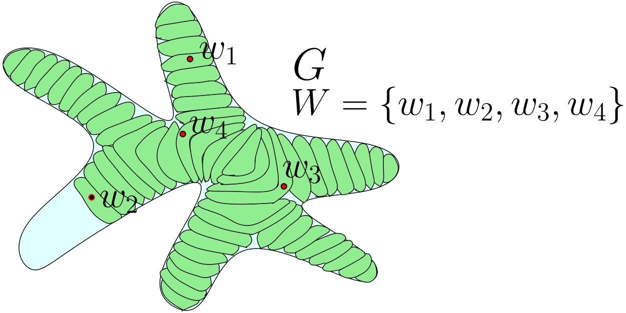

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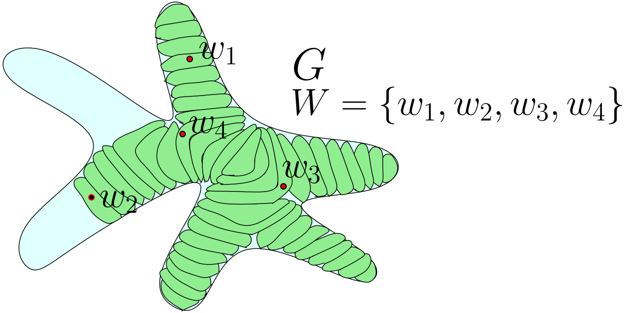

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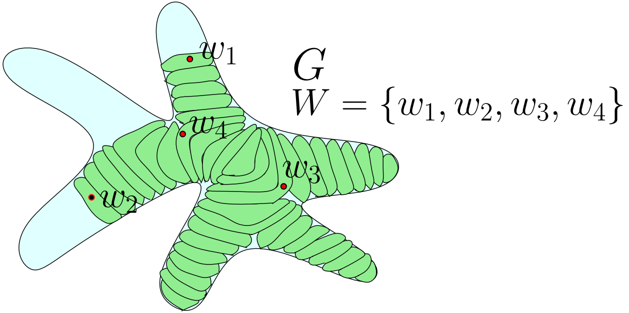

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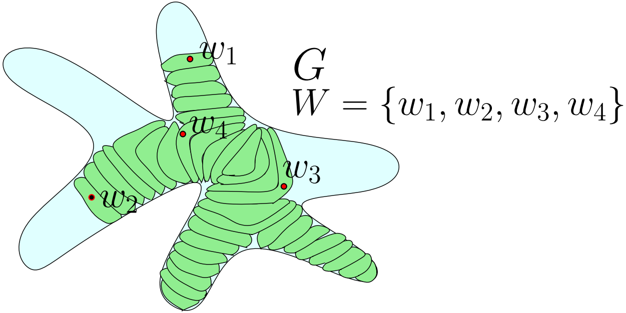

Improving a tree decomposition

Let W be a largest bag of T — SUBSET TREEWIDTH

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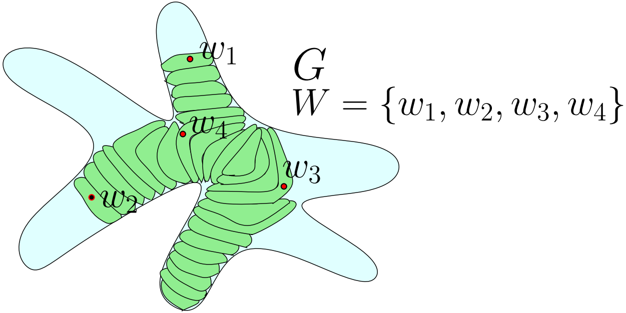

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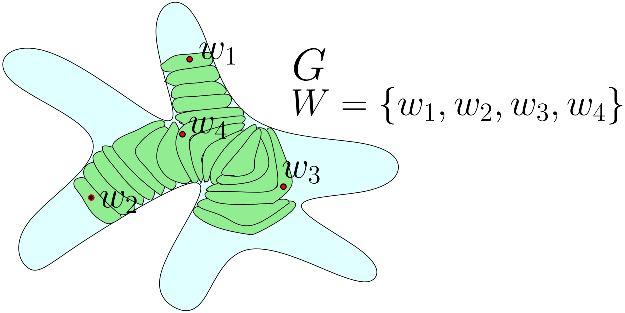

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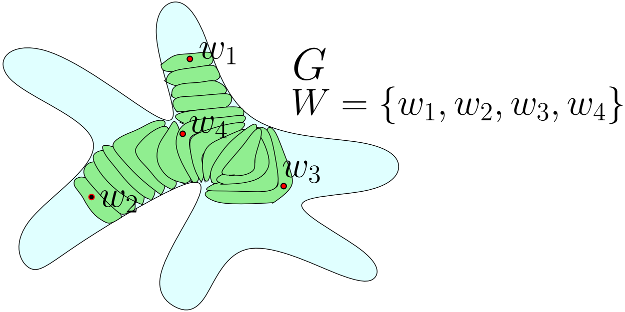

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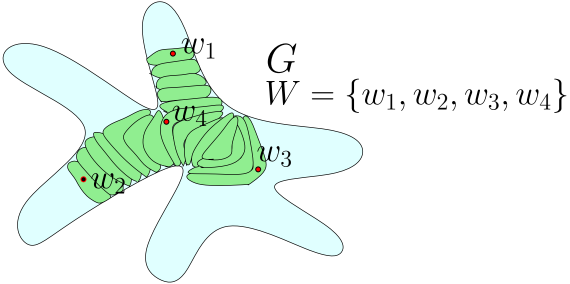

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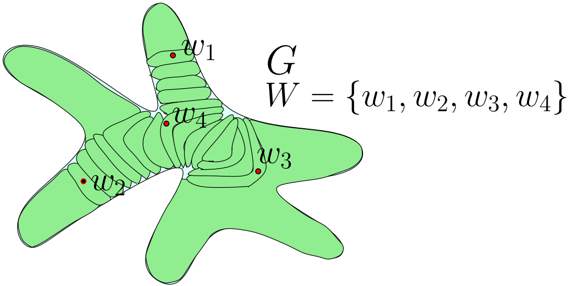
Big-leaf formulation:

- Find a tree decomposition of G whose internal bags have size $< |W|$ and cover W, but leaf bags can be arbitrarily large



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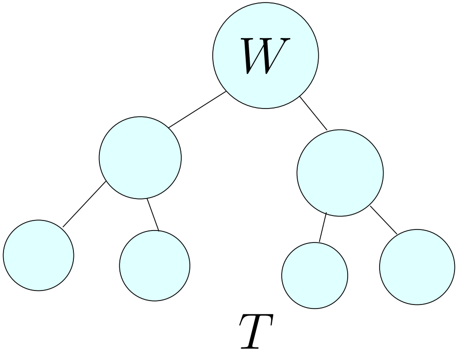

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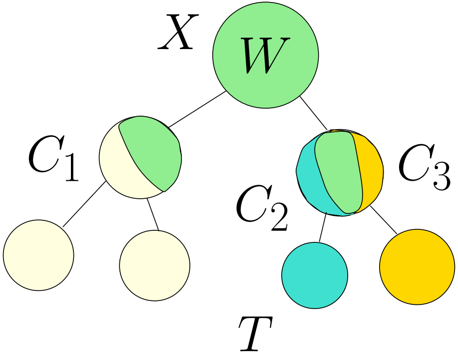

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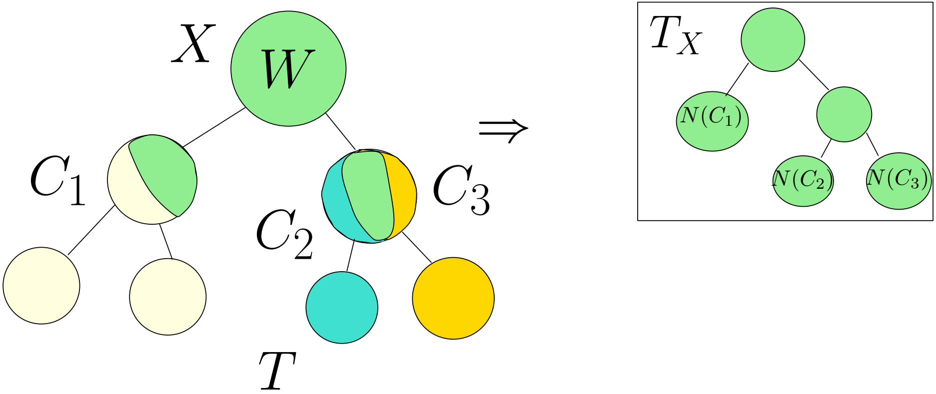

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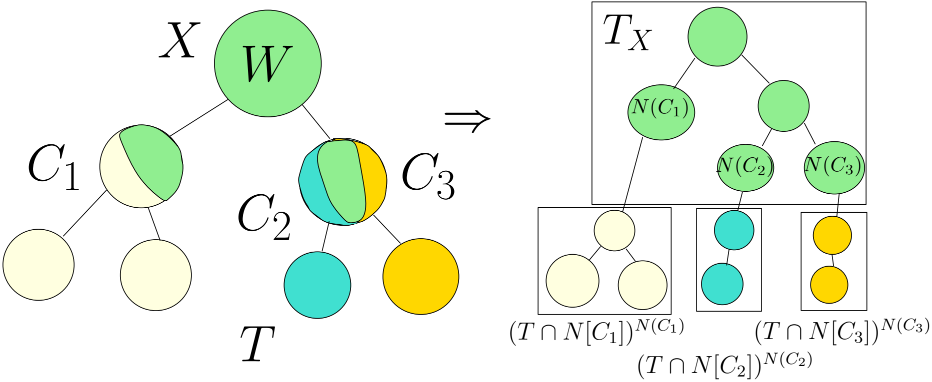

Constructing $(T \cap N[C_i])^{N(C_i)}$

Goal: For each component C_i of $G \setminus X$, construct a tree decomposition of $G[N[C_i]]$ so that $N(C_i)$ is in the root

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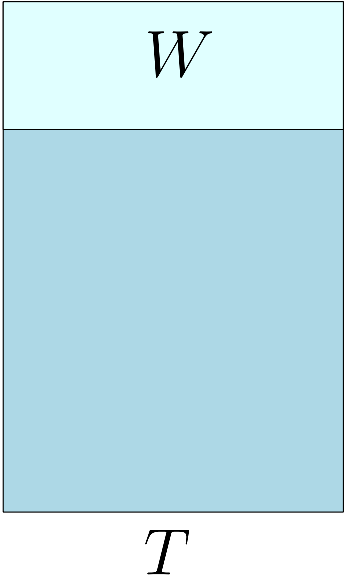

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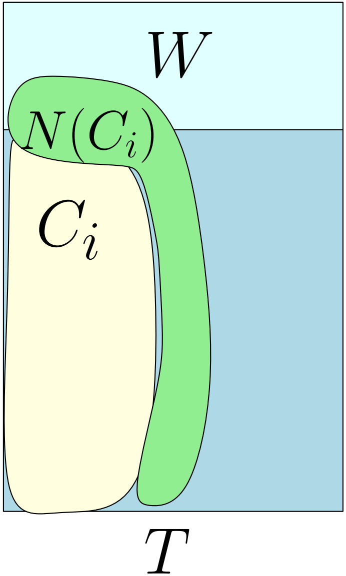

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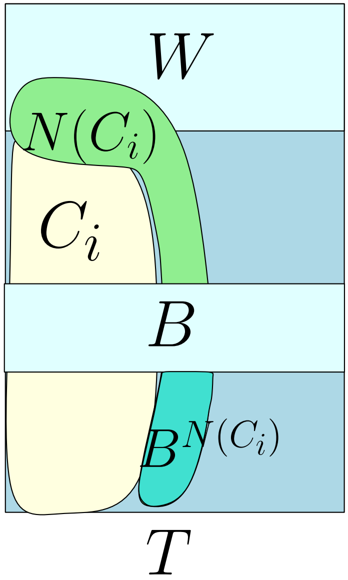

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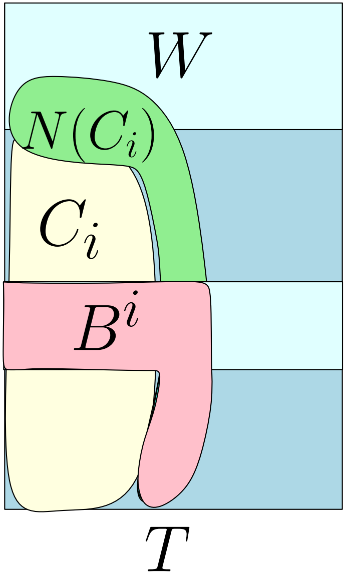

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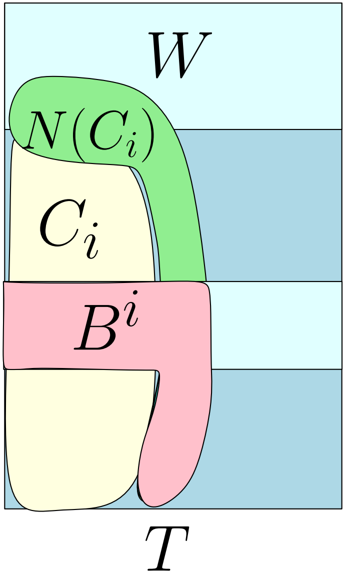

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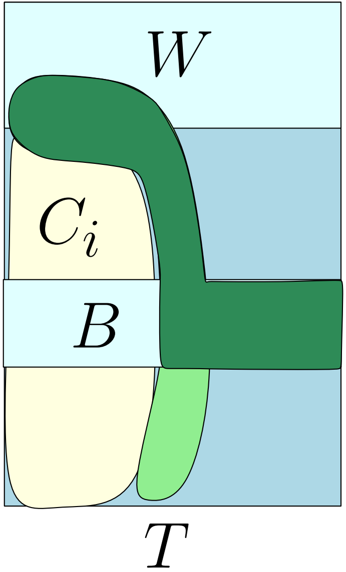

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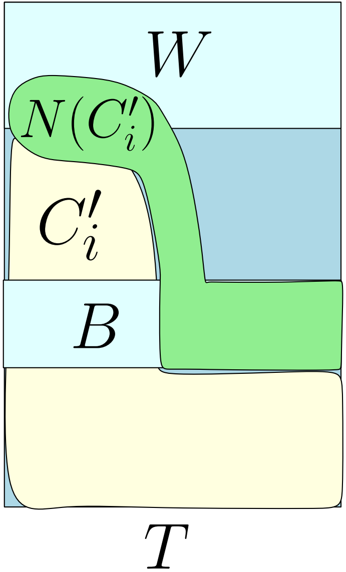

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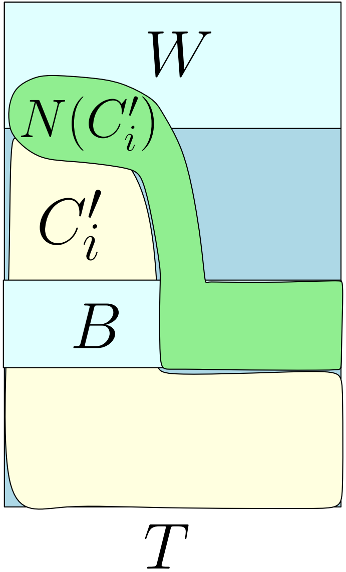
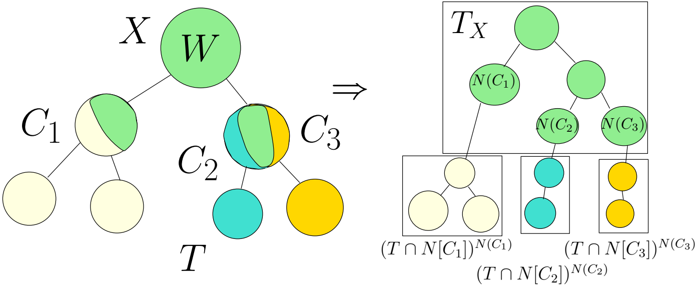
Decreases $|X|$



Result



- By repeatedly applying the pushing argument, we achieve:

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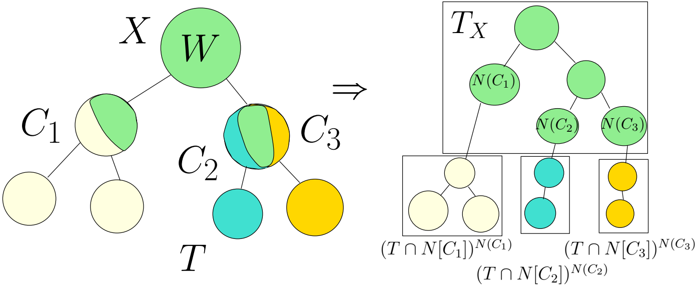

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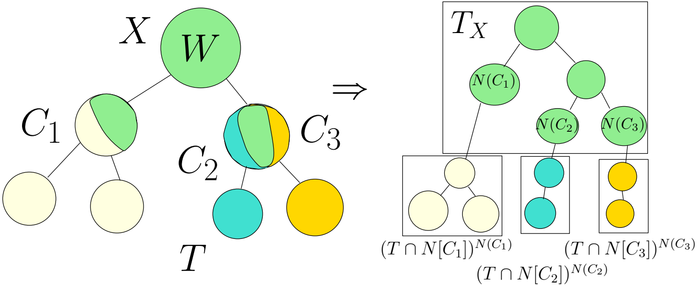

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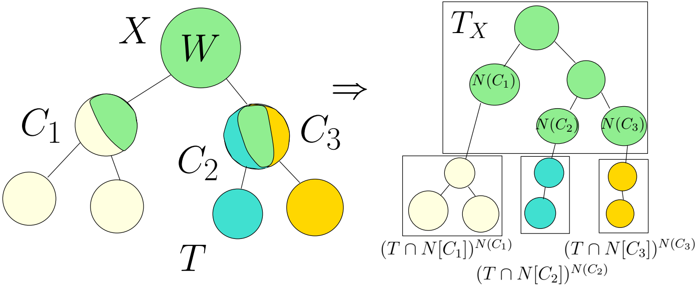

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- Proof idea generalization of proofs of existence of lean tree decompositions [Thomas '90, Bellenbaum & Diestel '02]

Subset treewidth for exact FPT algorithms

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SUBSET TREEWIDTH

Input: Graph G, integer k, set of vertices $W \subseteq V(G)$ with $|W| = k + 2$

Output: Set $X \subseteq V(G)$ with $W \subseteq X$ and tree decomposition of $\texttt{torso}(X)$ of width $\leq k$ or that the treewidth of G is $> k$

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Theorem

If there is an $f(k) \cdot n^{\mathcal{O}(1)}$ time algorithm for subset treewidth, then there is an $f(k) \cdot n^{\mathcal{O}(1)}$ time algorithm for treewidth with the same function f.

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$2^{\mathcal{O}(k^2)} n^2$ time algorithm for subset treewidth $\rightarrow$ $2^{\mathcal{O}(k^2)} n^4$ time algorithm for treewidth

How to solve subset treewidth?

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Techniques:

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1. Branching on important separators [Marx'06]

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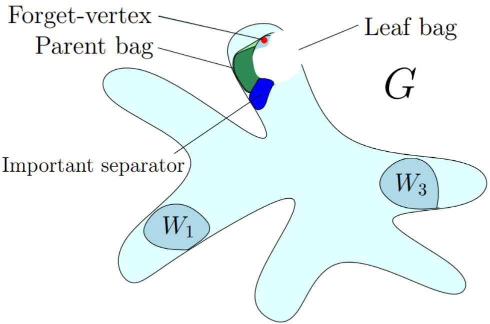

Local improvement in dynamic treewidth

joint work with Konrad Majewski, Wojciech Nadara, Michał Pilipczuk & Marek Sokołowski

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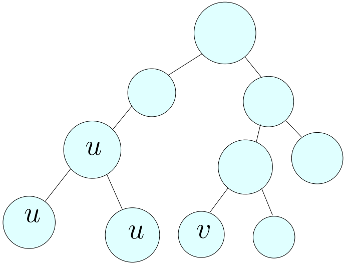

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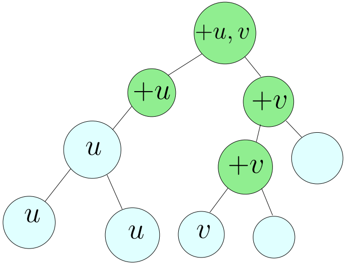

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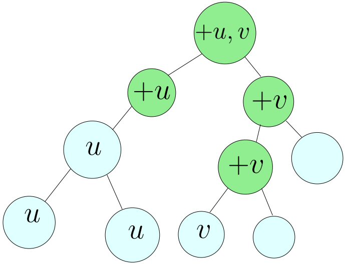

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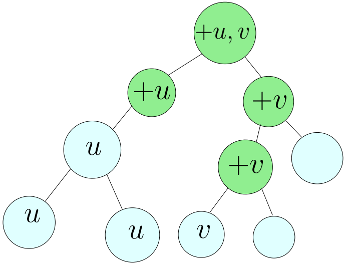

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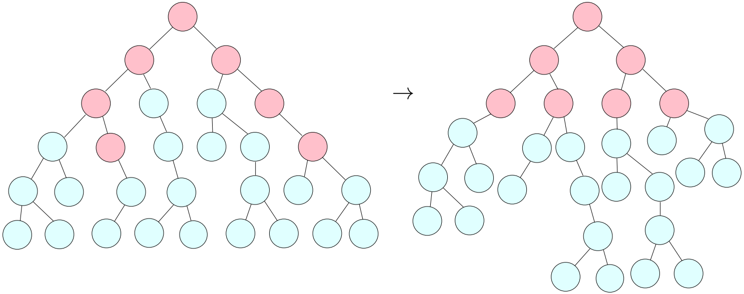

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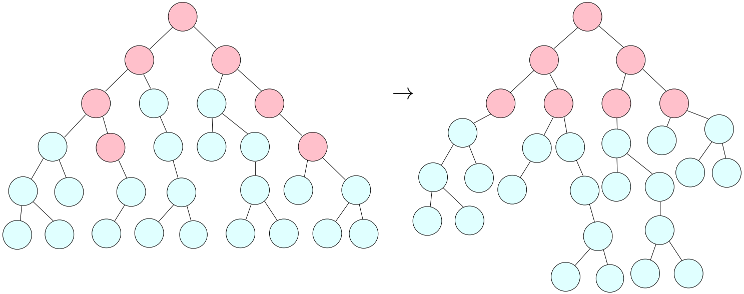

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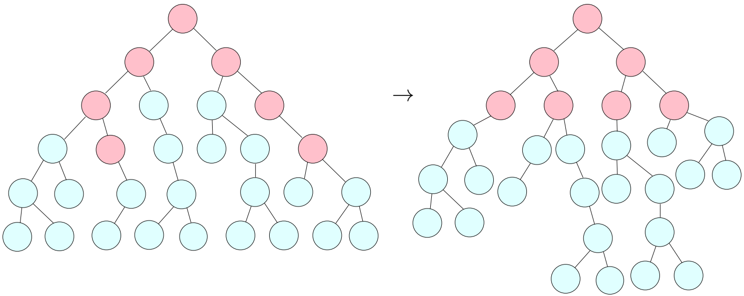
- Builds on subset treewidth, log-depth decompositions [Bodlaender & Hagerup '98], and the "dealternation lemma" [Bojańczyk & Pilipczuk '22]



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- Introduced in [K. '21] for 2-approximation in $2^{\mathcal{O}(k)} n$ time
- Generalized in [K. & Lokshtanov '23] for exact in $2^{\mathcal{O}(k^2)} n^4$ time and $(1+\varepsilon)$-approximation in $k^{\mathcal{O}(k/\varepsilon)} n^4$ time
- Used in [K., Majewski, Nadara, Pilipczuk & Sokołowski '23] for fully dynamic treewidth in $f(k) \cdot n^{o(1)}$ amortized update time

Open problems:

- Prove $2^{\Omega(k)}$ lower bound for treewidth under ETH ($2^{\Omega(\sqrt{k})}$ known)
- Treewidth 1.9-approximation in $2^{\mathcal{O}(k)} n^{\mathcal{O}(1)}$ time?
- Dynamic treewidth in amortized $f(k) \cdot \text{polylog}(n)$ time?

Thank you!