Fast FPT-Approximation of Branchwidth

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University of Bergen, Norway

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- Framework for designing FPT 2-approximation algorithms for branchwidth of symmetric submodular functions

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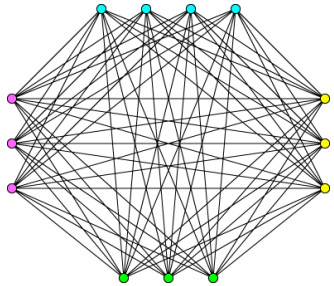

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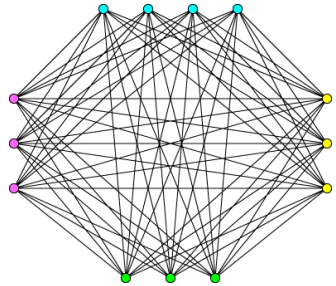

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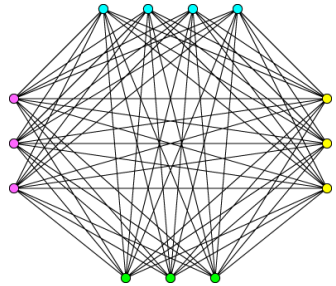

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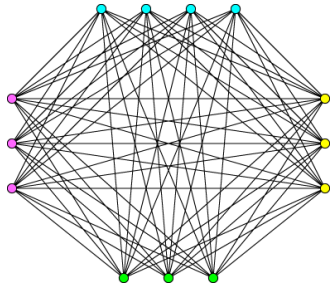

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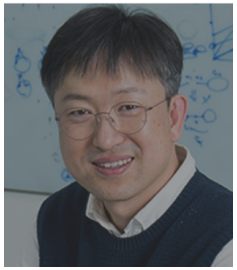

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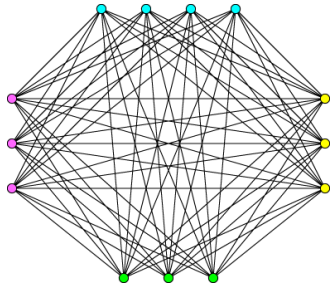

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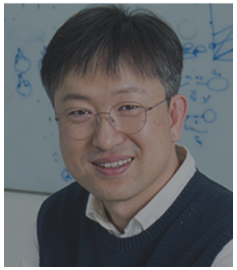

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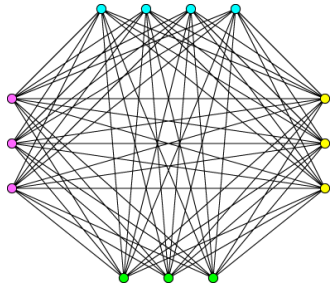

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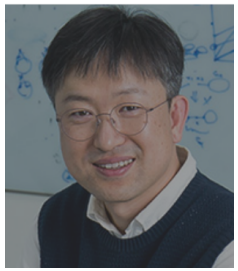

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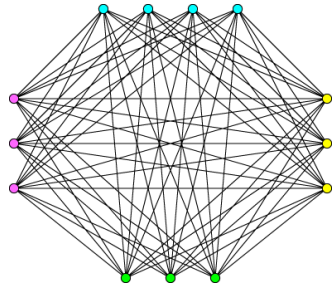

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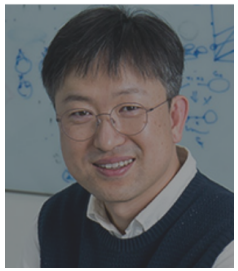

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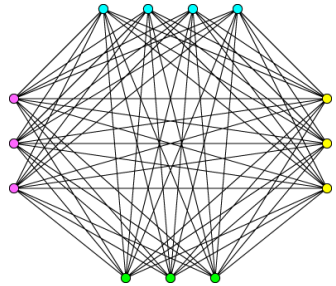

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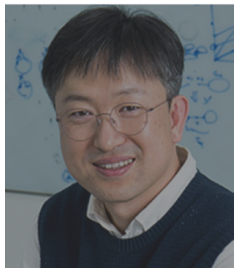

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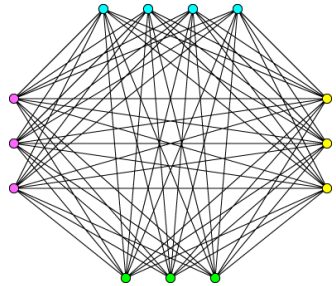

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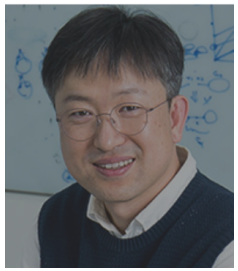

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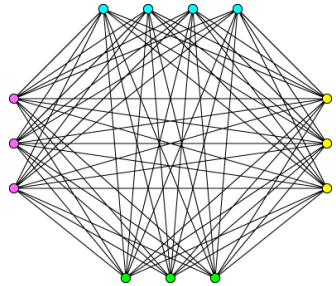

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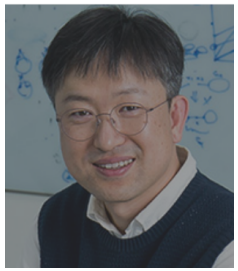

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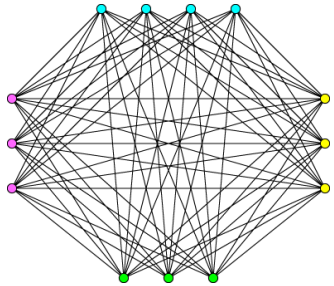

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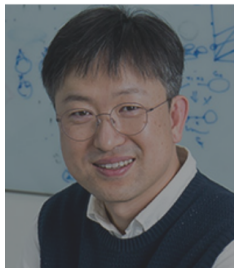

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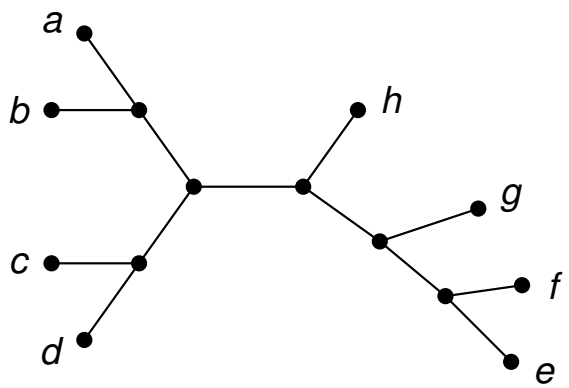

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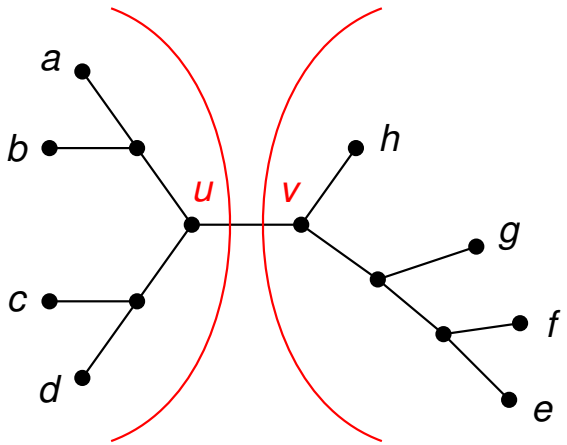

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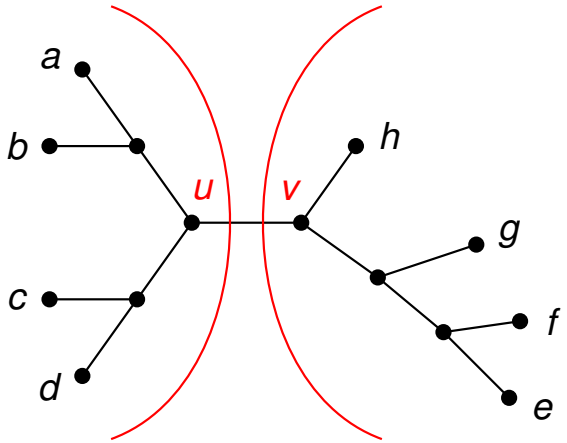

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- Also carving-width, matroid branchwidth, rankwidth in different fields...

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  - Apply with treewidth approximation to get $2^{\mathcal{O}(k)}n$ time algorithm

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  - For rankwidth $t(k) = 2^{2^{\mathcal{O}(k)}}$, for graph branchwidth $t(k) = 2^{\mathcal{O}(k)}$

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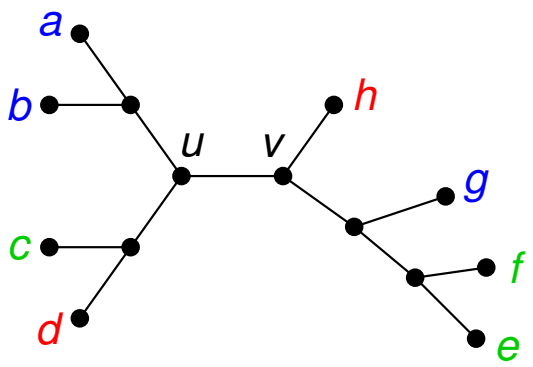

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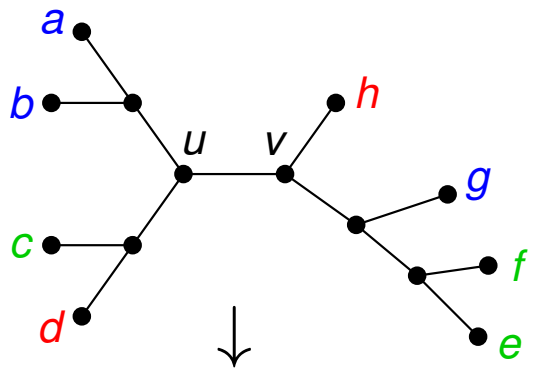

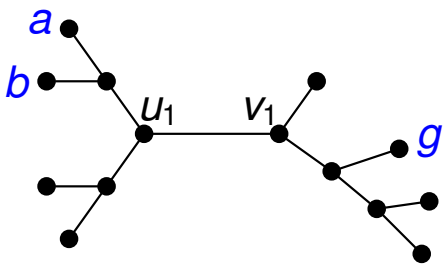

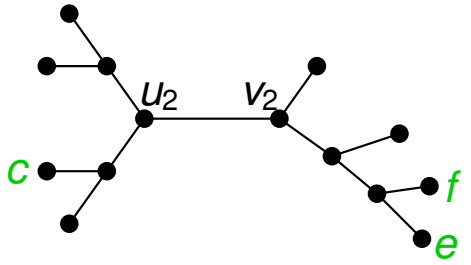

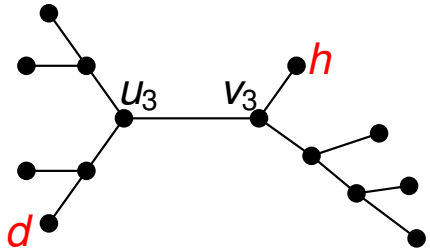

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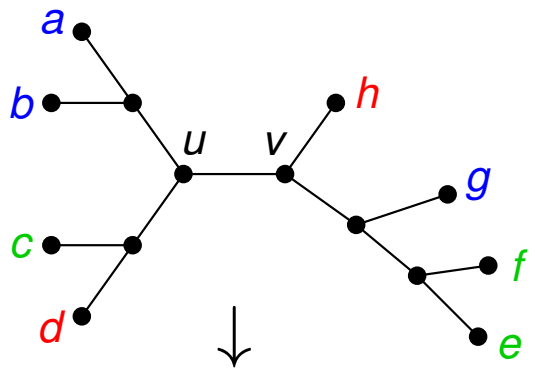

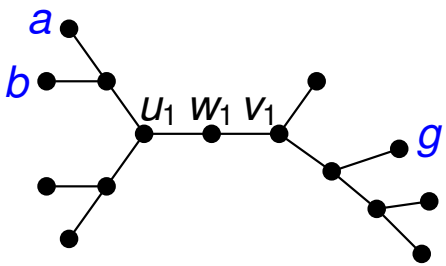

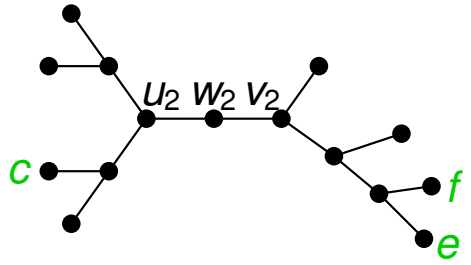

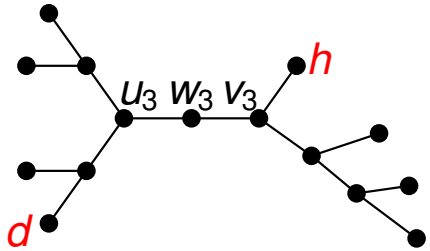

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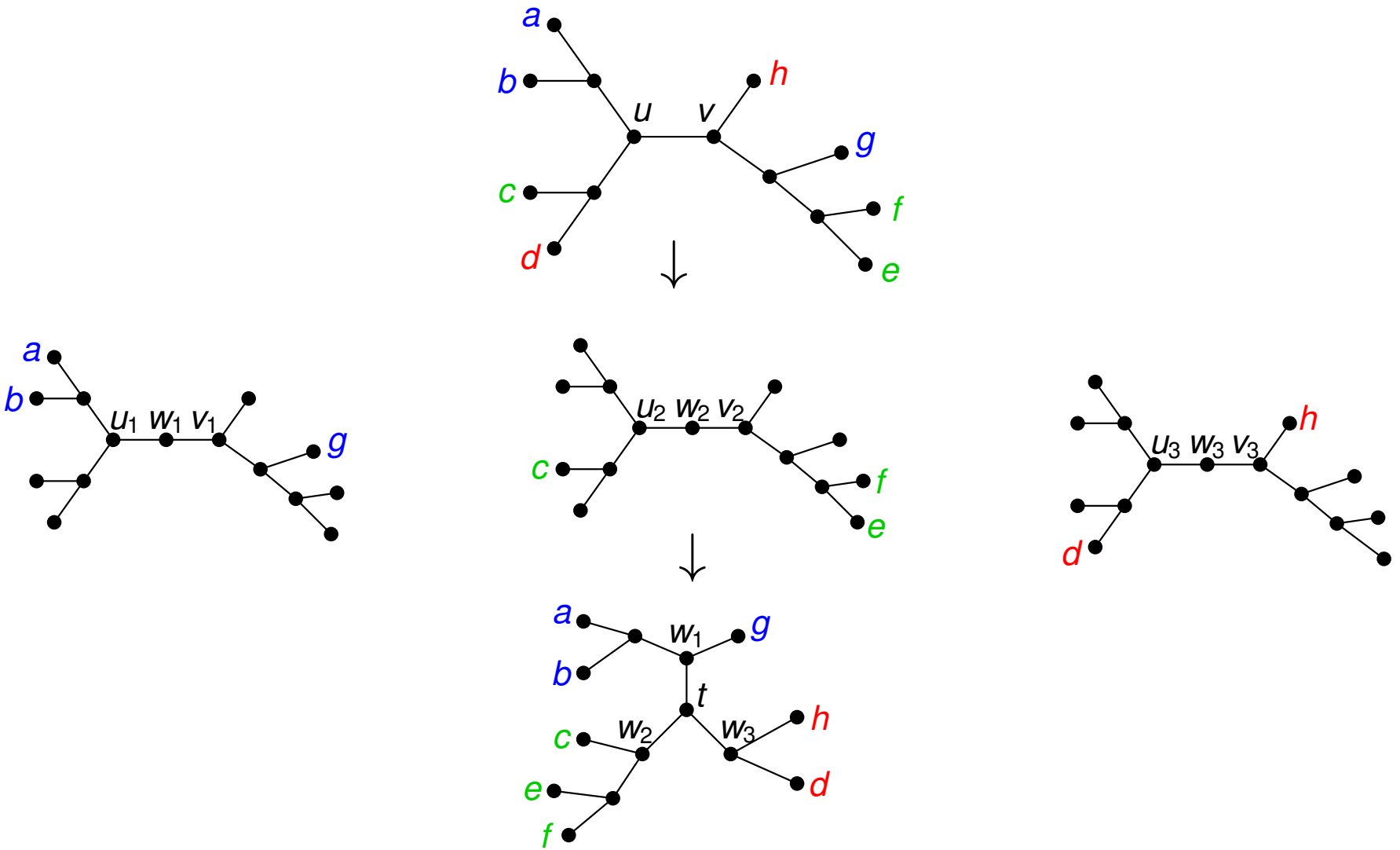

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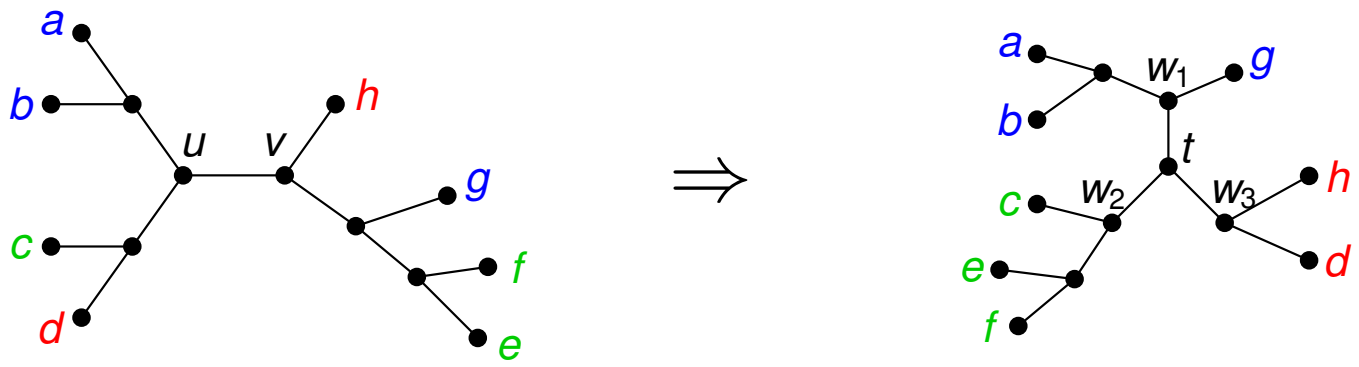

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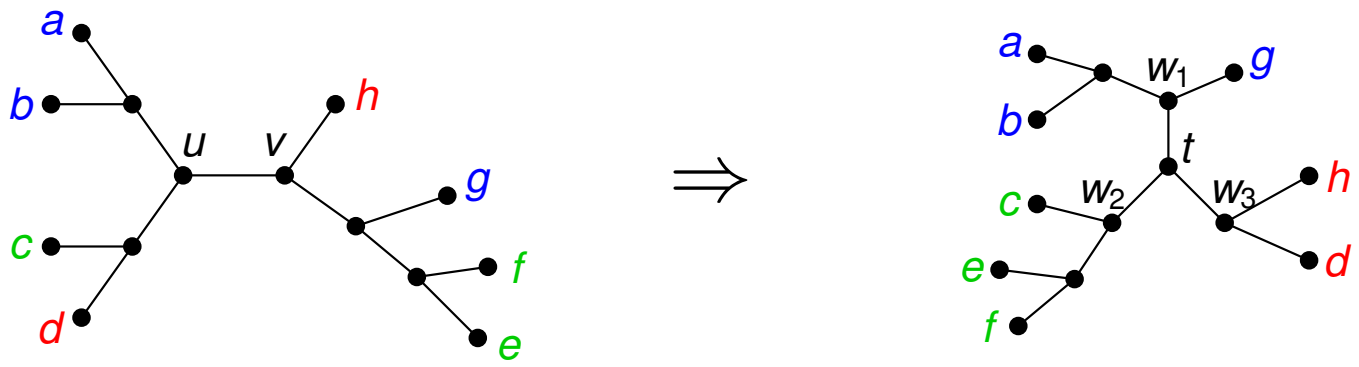

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- Too slow! Goal is linear in n

Linear-time Algorithm

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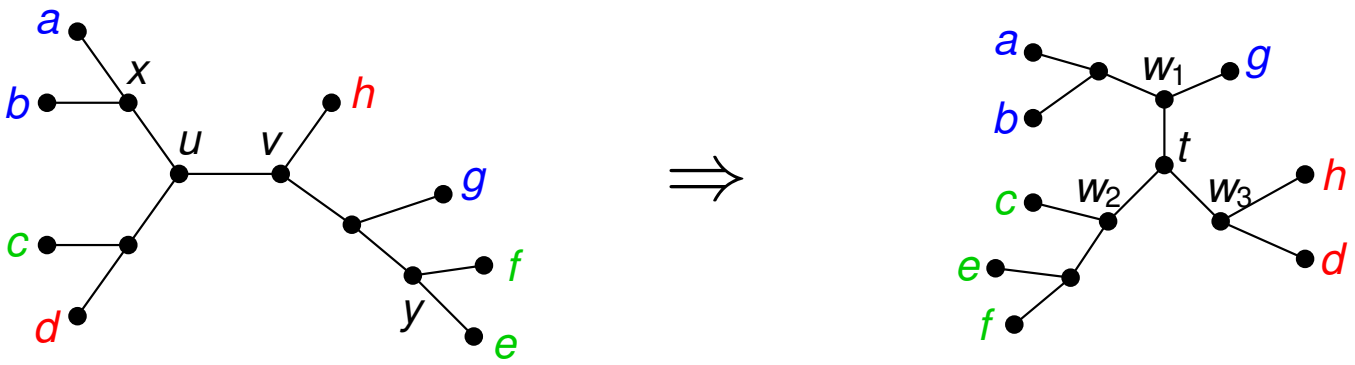

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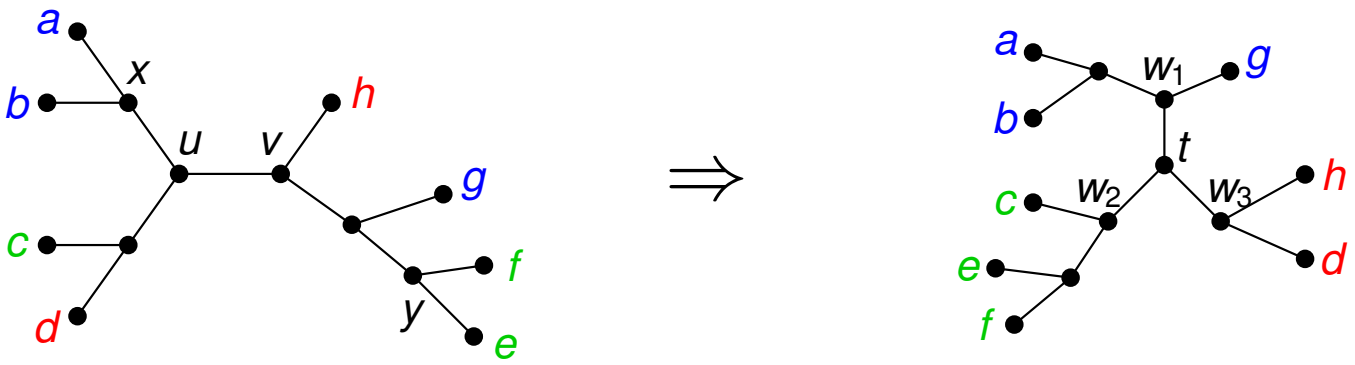

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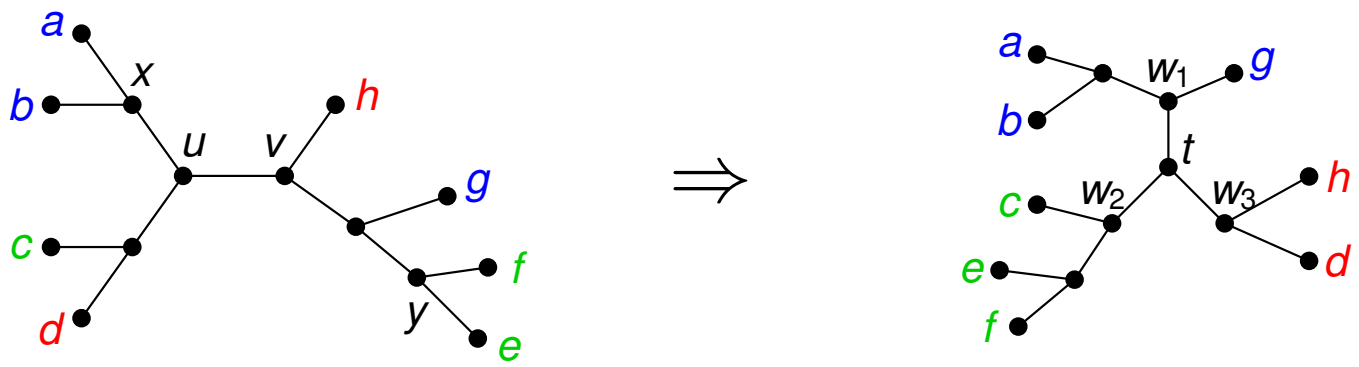

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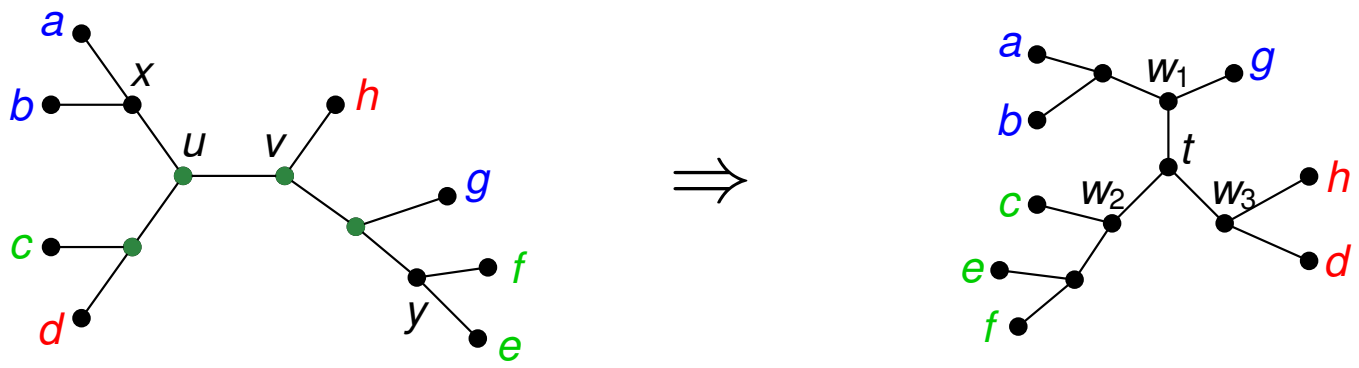

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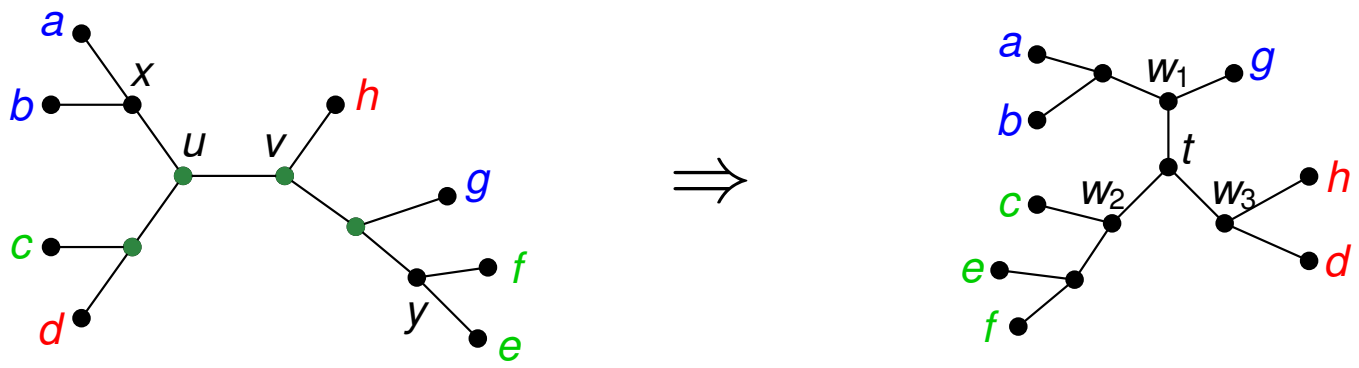

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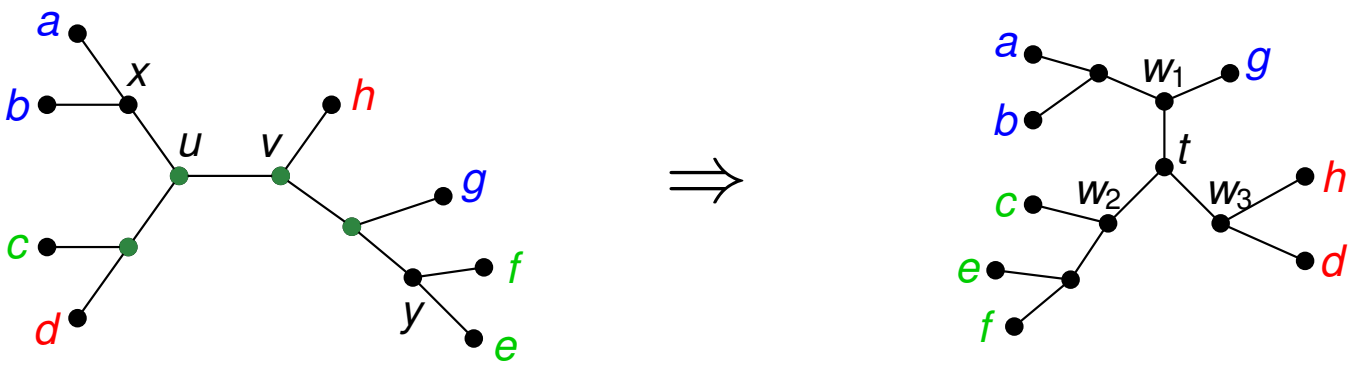

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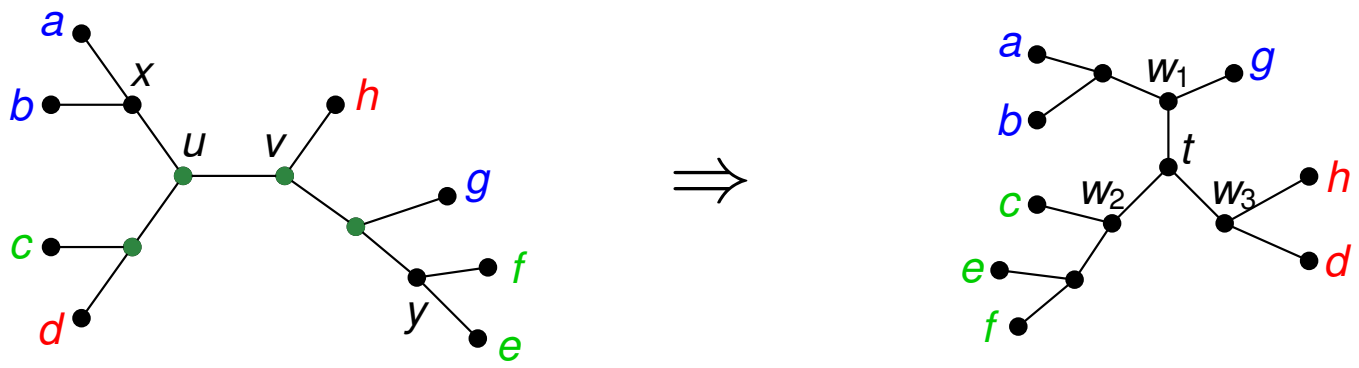

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- Walk over the decomposition and refine whenever seeing edge uv with $f(uv) = k$

The end

Thanks for watching!

Paper: https://arxiv.org/abs/2111.03492

Slides: https://tuukkakorhonen.com