Fast FPT 2-Approximation Algorithms for Width Parameters

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FPT-Approximation of Treewidth

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Robertson & Seymour, Graph Minors XIII. ('86)

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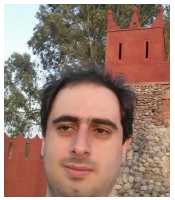


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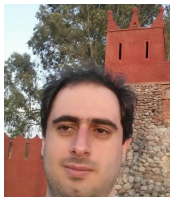


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- Wu, Austrin, Pitassi, Liu'14: No polynomial-time constant-factor approximation assuming the small set expansion hypothesis

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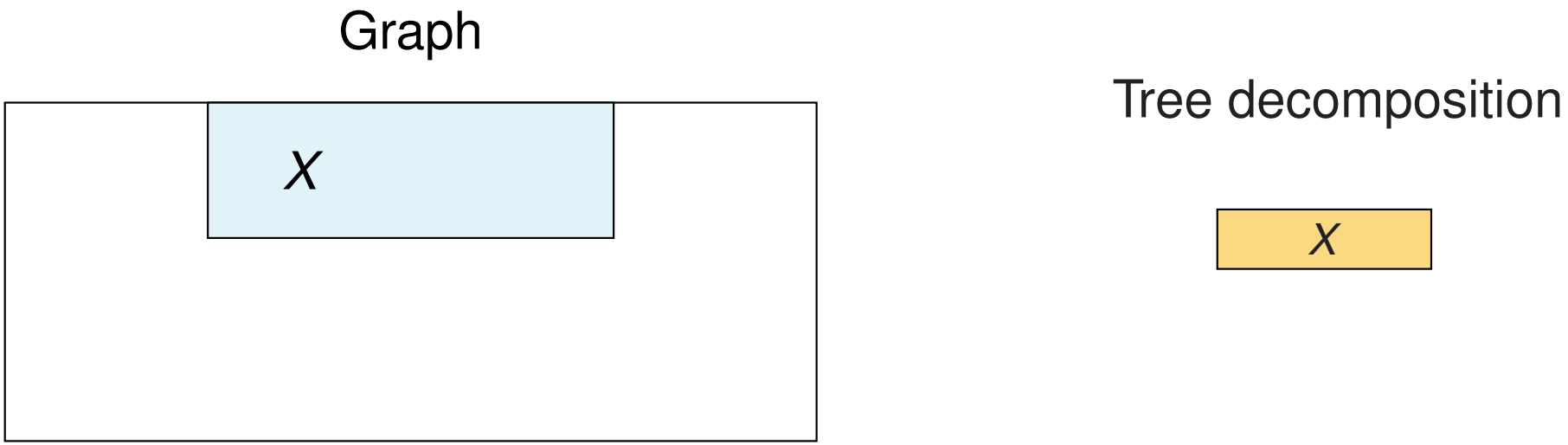

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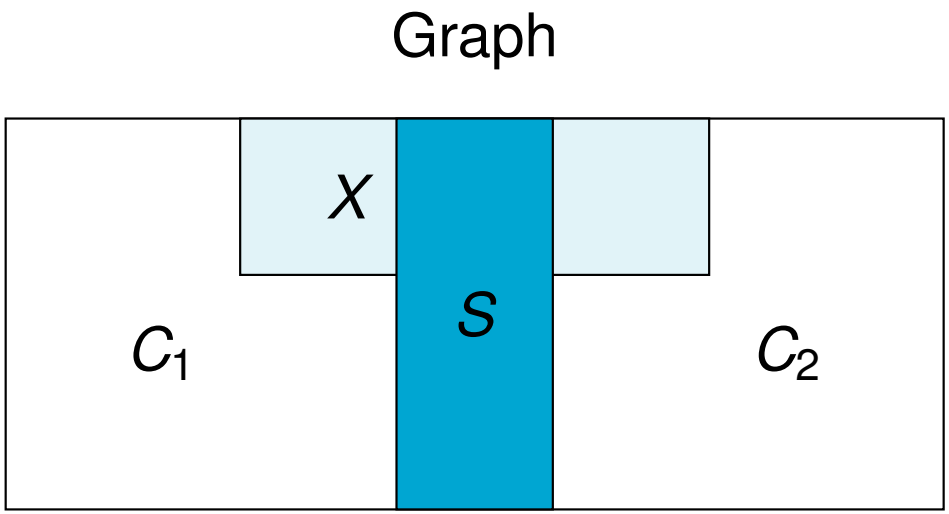

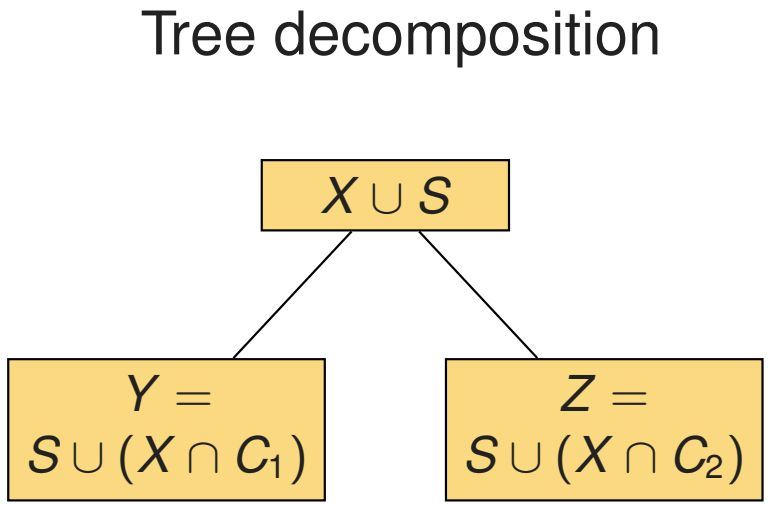

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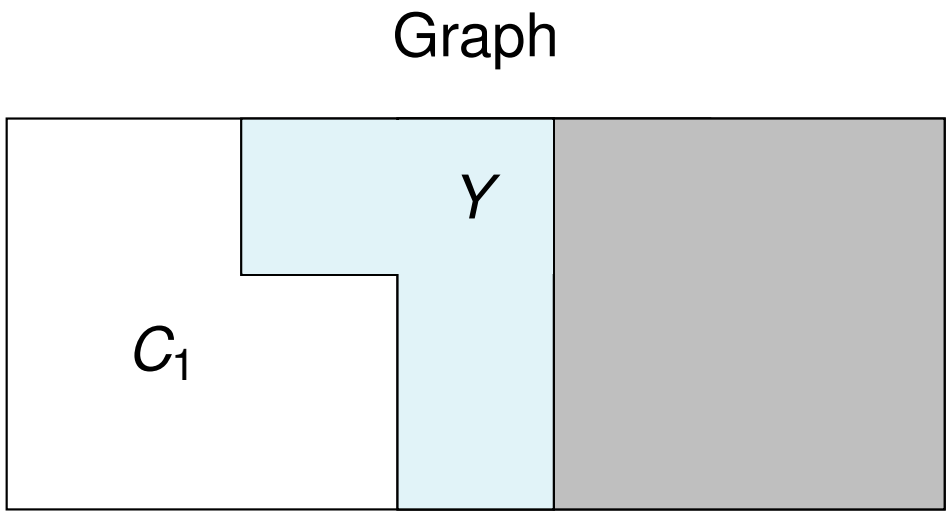

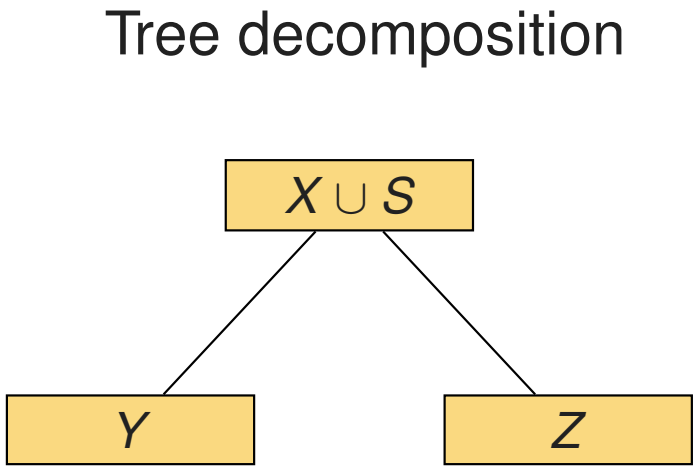

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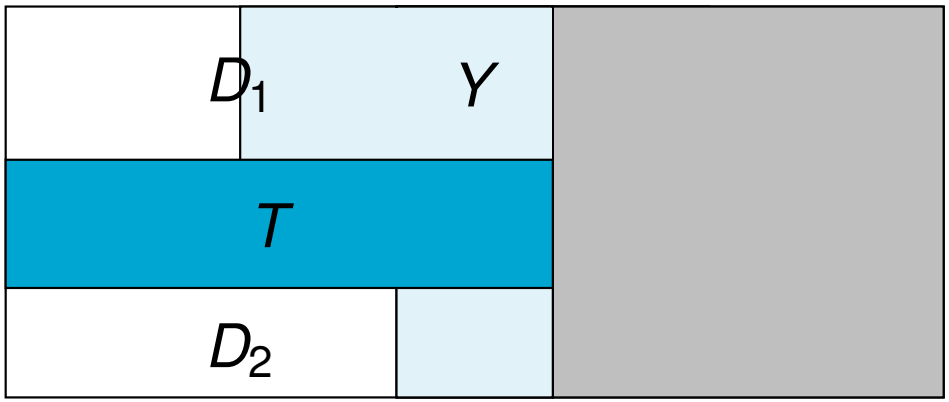

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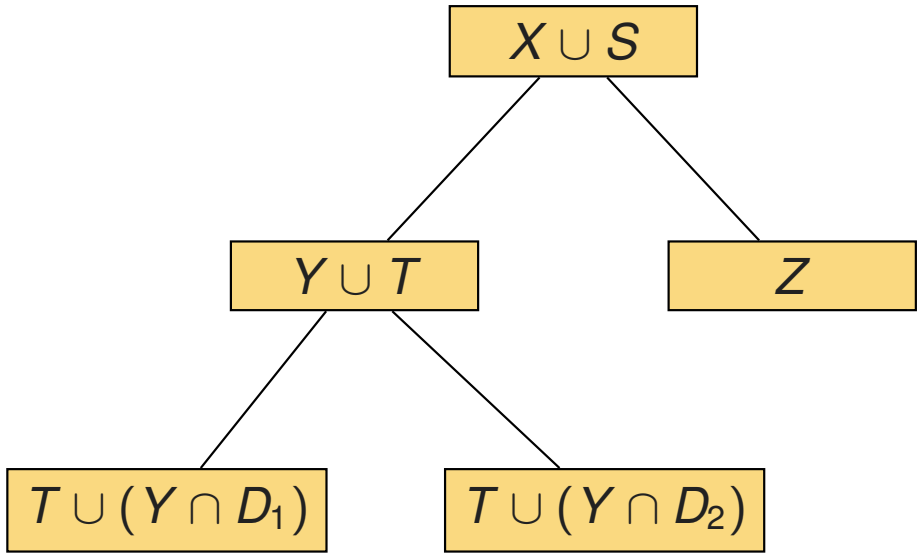
Tree decomposition



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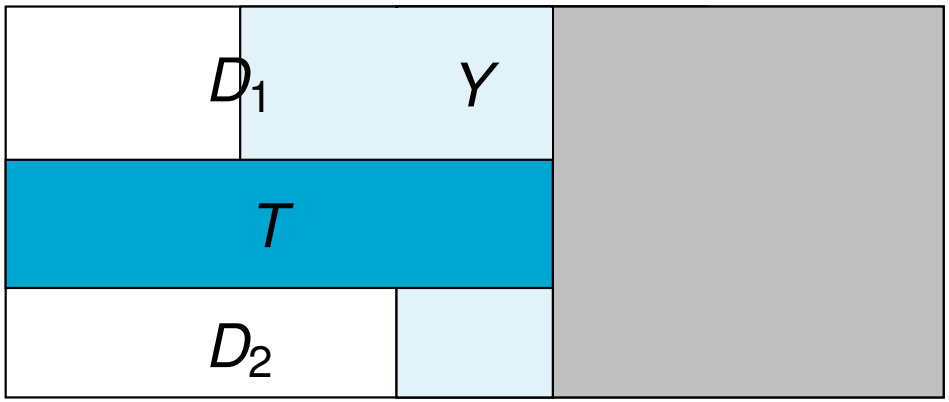

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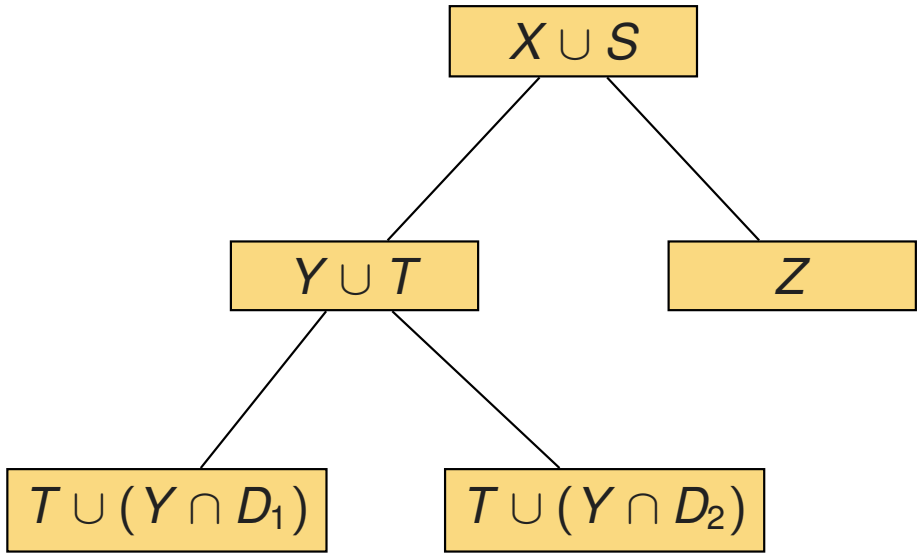

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Continue recursively...

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Also...

[Lagergren'90], [Feige, Hajiaghayi & Lee'05], [Amir'10],
[Fomin, Lokshtanov, Pilipczuk, Saurabh & Wrochna'17], [Belbasi & Fürer'21]

Branchwidth of Connectivity function

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- Let V be a set and $f : 2^V \to \mathbb{Z}_{\geq 0}$ a connectivity function:
  - Symmetric: For any $A \subseteq V$, it holds that $f(A) = f(\overline{A})$, where $\overline{A} = V \setminus A$
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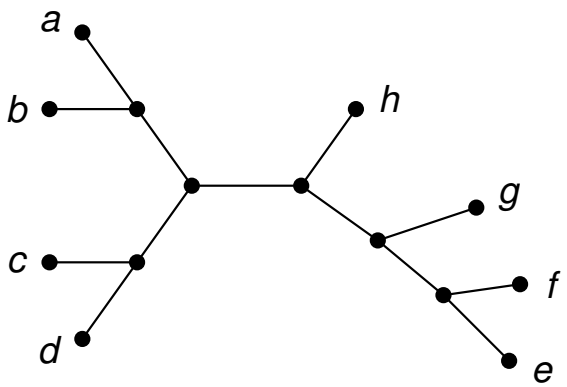
- Branch decomposition of f is a cubic tree whose leaves are the elements of V
  - Example with $V = \{a, b, c, d, e, f, g, h\}$:



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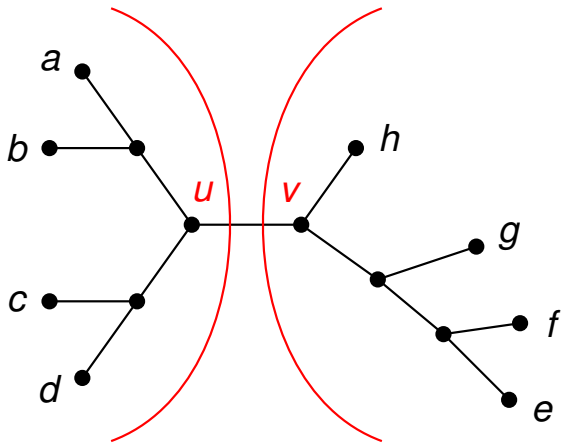

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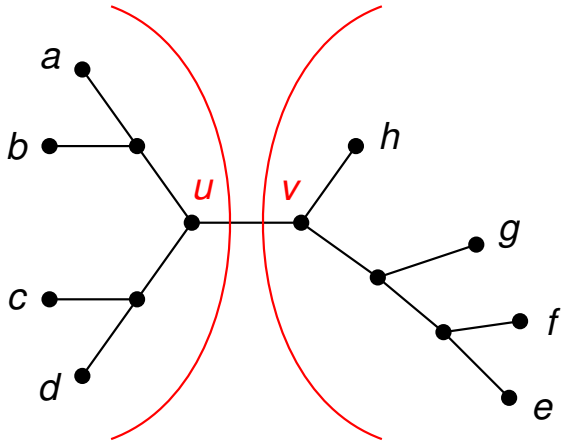

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- The branchwidth of f is minimum width of a branch decomposition of f

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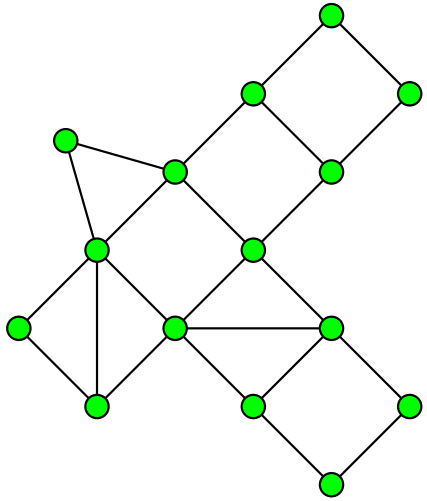

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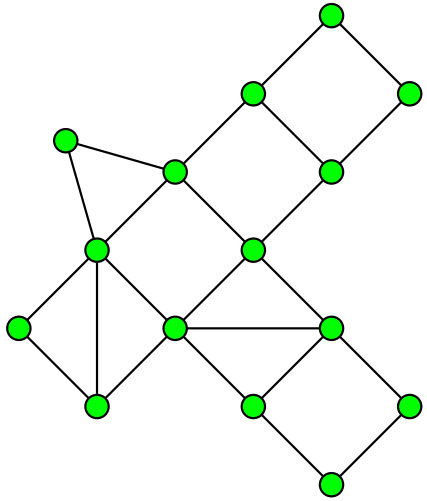

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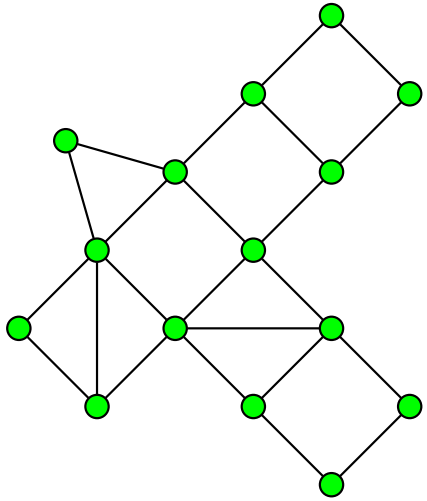

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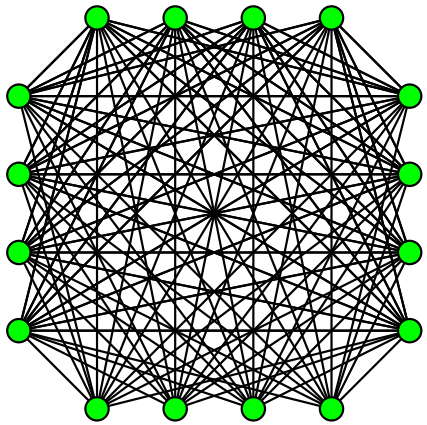

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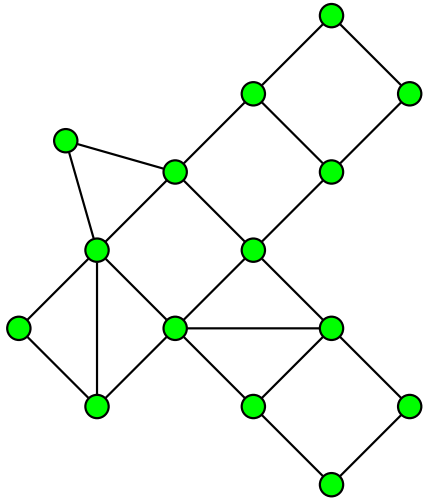

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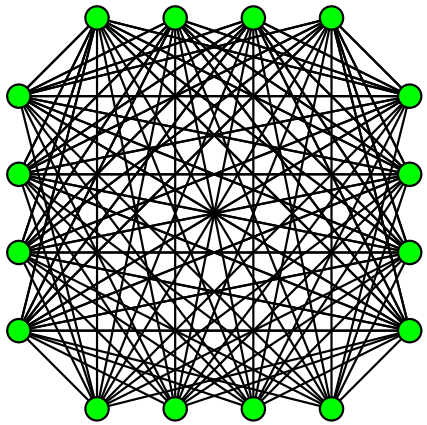

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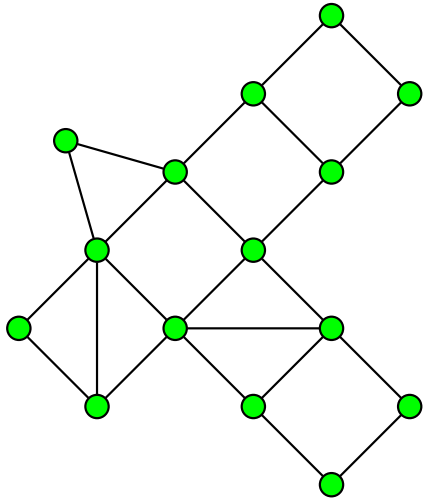

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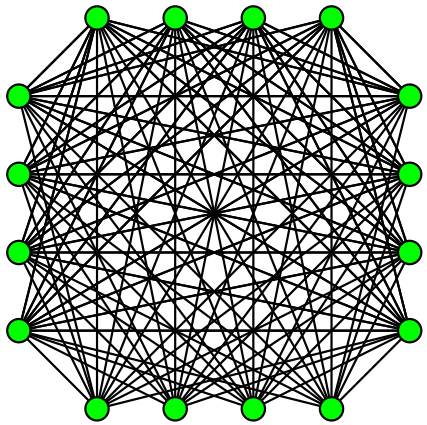

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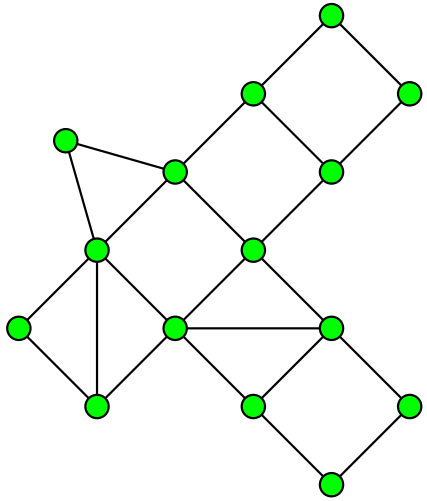

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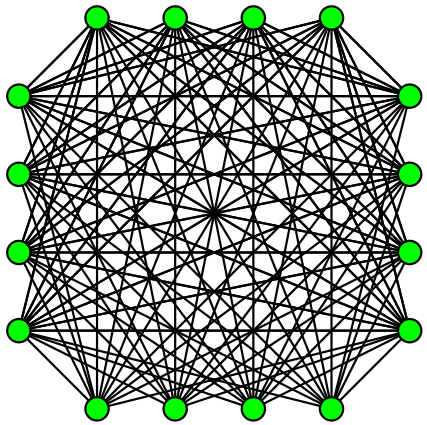

- Also carving-width, matroid branchwidth, rankwidth over different fields...

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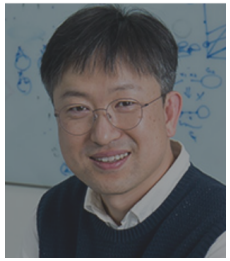
Oum & Seymour ’06:

- 3-approximation of branchwidth of connectivity function in time $8^k n^{\mathcal{O}(1)}$

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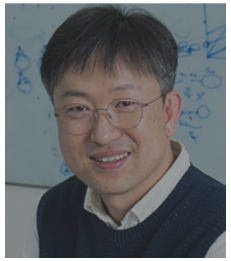


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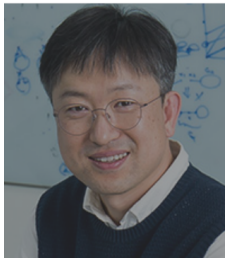


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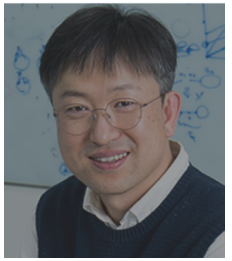


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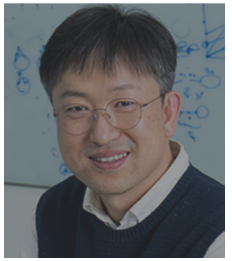


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- Similar top-down construction as Robertson-Seymour algorithm

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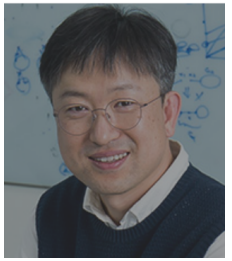


- 3-approximation of branchwidth of connectivity function in time $8^k n^{\mathcal{O}(1)}$

$\Rightarrow$ 3-approximation of rankwidth in time $\mathcal{O}(8^k n^9 \log n)$

$\Rightarrow$ 2^{3k+2}-approximation of cliquewidth in time $\mathcal{O}(8^k n^9 \log n)$

$\Rightarrow$ $f(k)n^9 \log n$ time algorithms parameterized by cliquewidth/rankwidth

- Similar top-down construction as Robertson-Seymour algorithm
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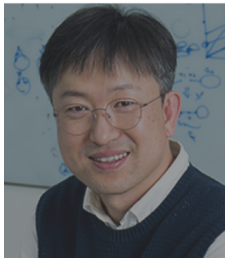


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- Further improved to $f(k)n^3$ time exact rankwidth by [Hlinený & Oum'08]

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- Instead of top-down construction, iteratively improve the decomposition
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- Interplay of improvement operations and dynamic programming
  - Improvements in time complexity both as a function of n and as a function of k

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- For rankwidth: $f(k) \cdot n$ time c-approximate compression algorithm $\rightarrow 2^{\mathcal{O}(k)} \cdot f(k) \cdot n^2$ time c-approximation algorithm

Plan

- 2-Approximating branchwidth of connectivity functions
  1. Combinatorial framework
  2. Algorithmic framework

Combinatorial Framework

General idea

- Setting:
  - Let $f : 2^V \to \mathbb{Z}_{\geq 0}$ be a connectivity function
  - We have a branch decomposition T of f of width k
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- Strategy:
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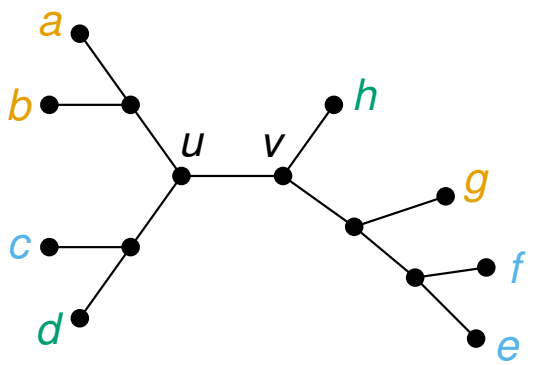
$\Rightarrow$ after at most n refinement operations, $\text{w}(T)$ decreases

Refinement operation

Specified by 4-tuple (uv, C_1, C_2, C_3), where $uv \in E(T)$ and (C_1, C_2, C_3) tripartition of V

Example with $(uv, C_1, C_2, C_3) = (uv, \{a, b, g\}, \{c, e, f\}, \{d, h\})$



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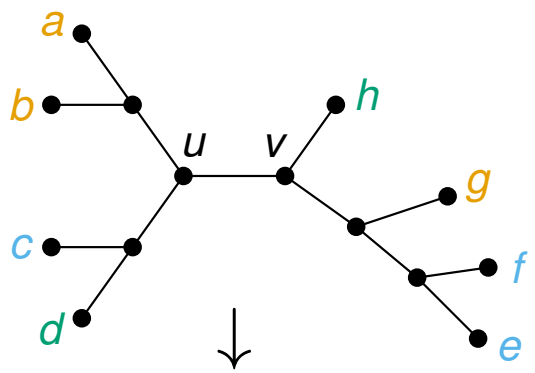

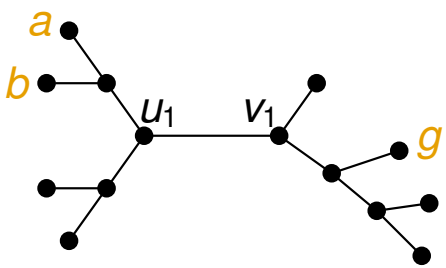

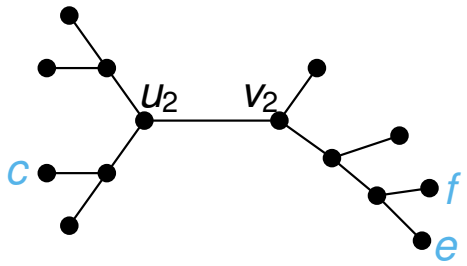

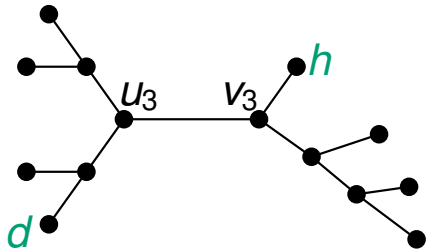

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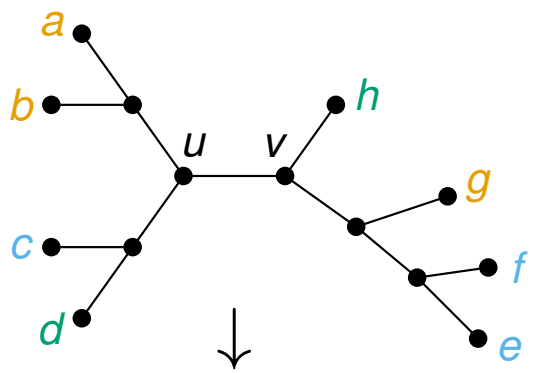

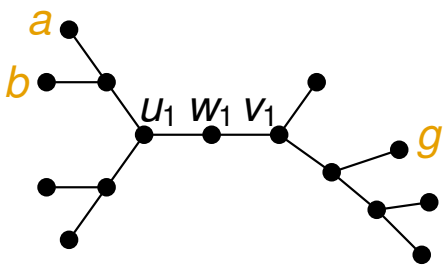

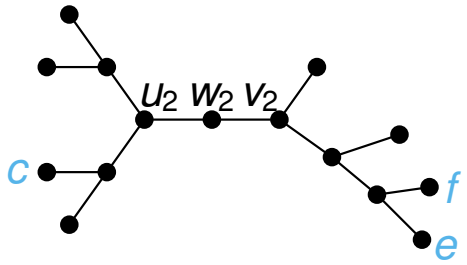

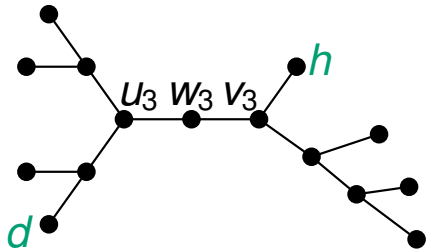

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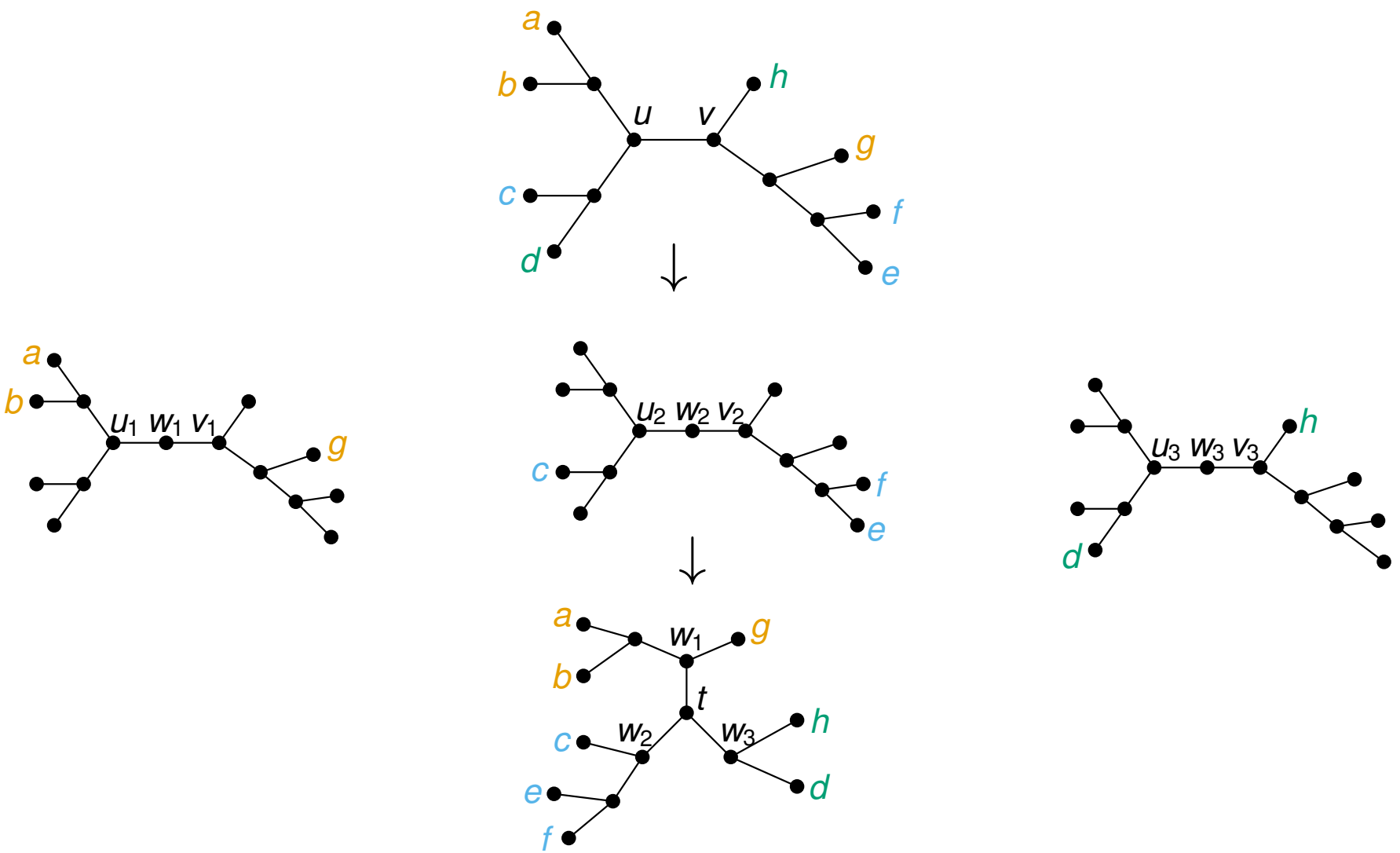
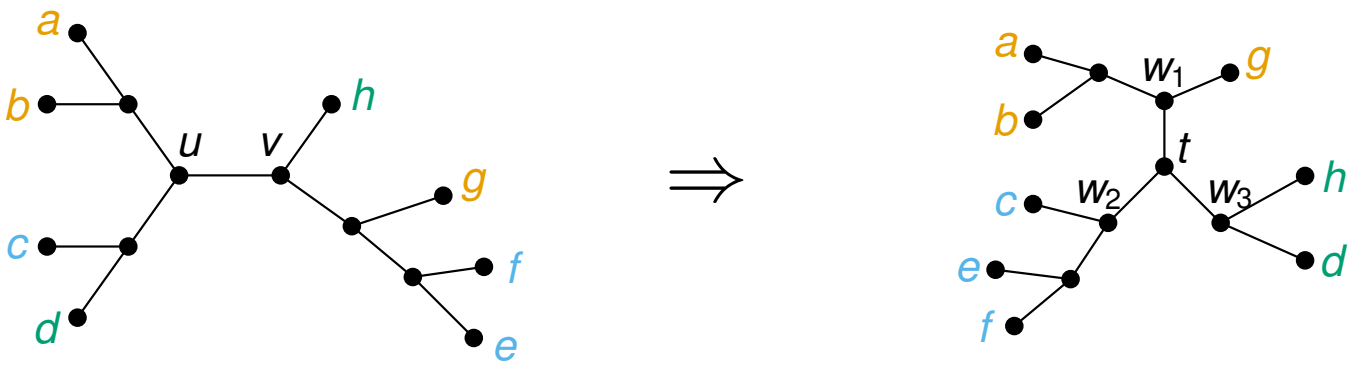


Observations on Refinement

Example with $(r, C_1, C_2, C_3) = (uv, \{a, b, g\}, \{c, e, f\}, \{d, h\})$



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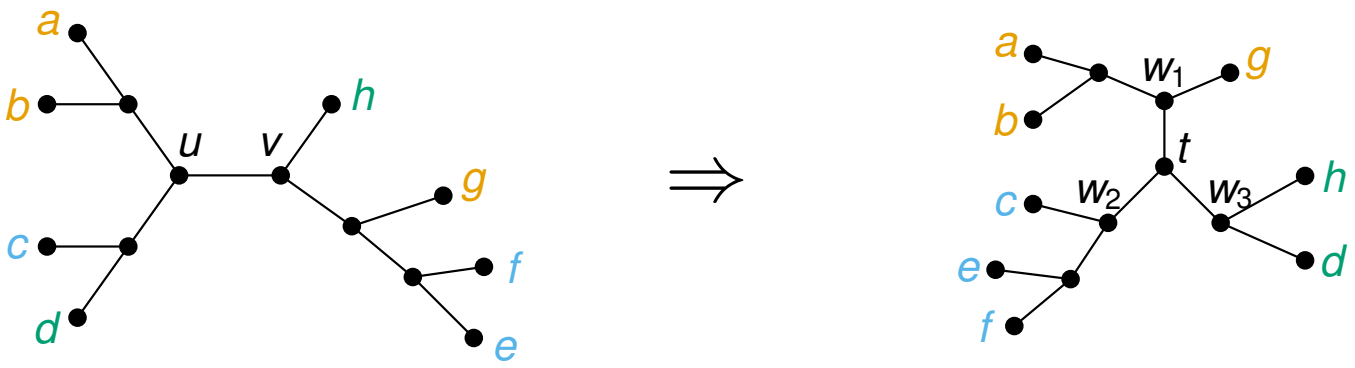

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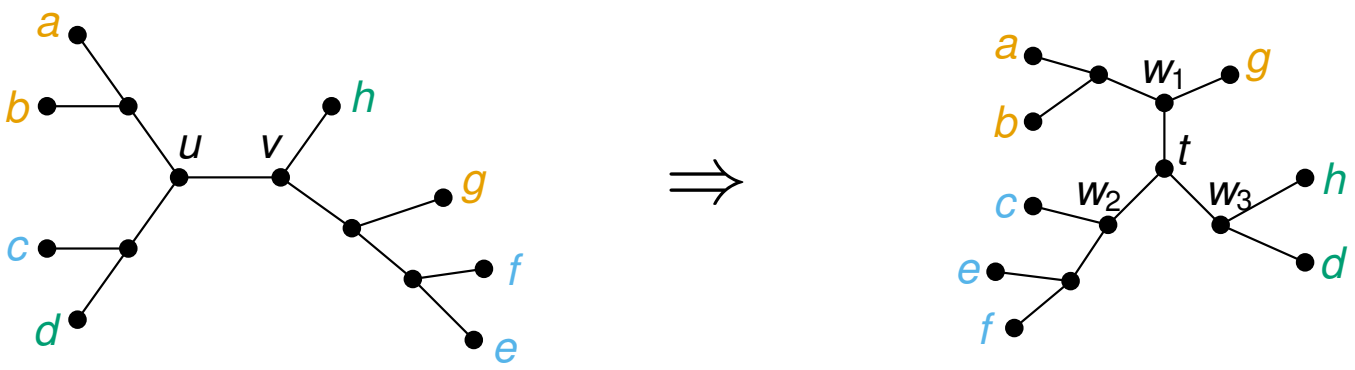

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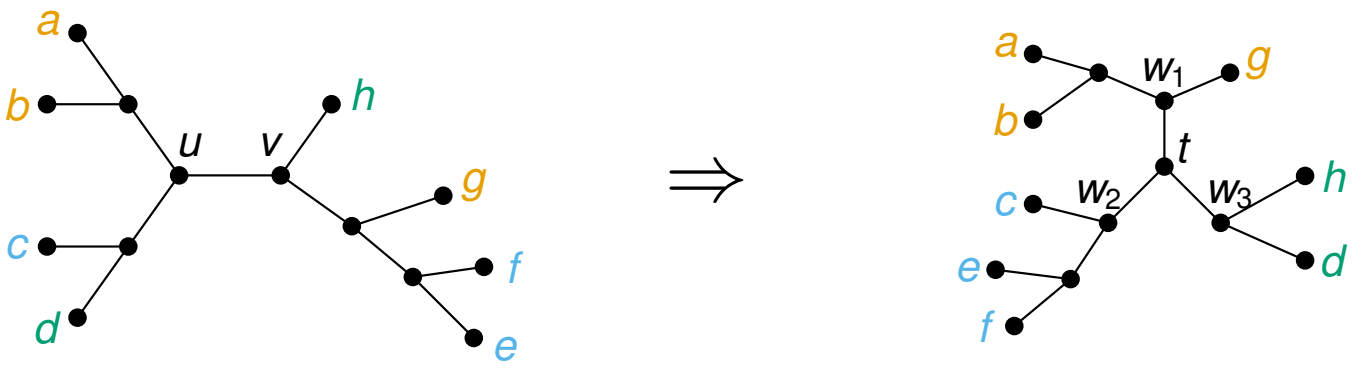

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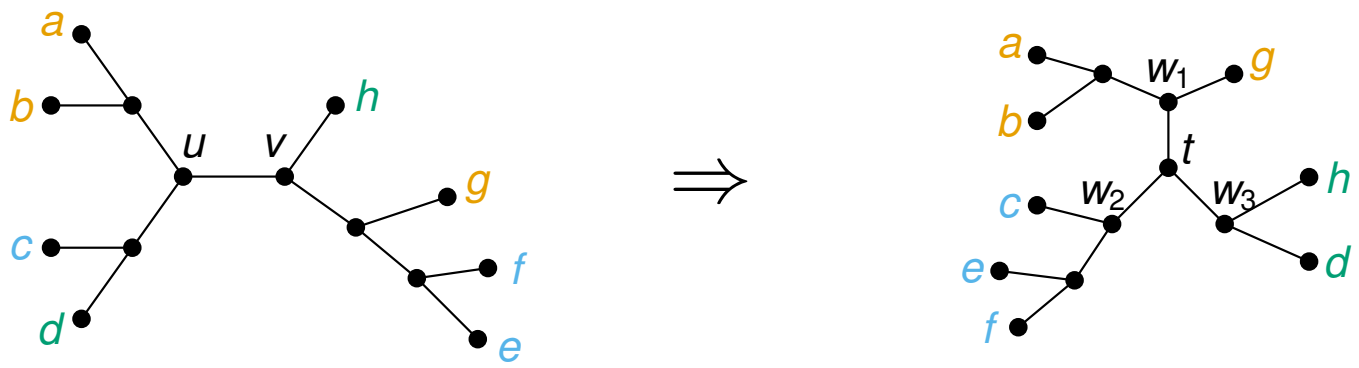
- Observation 2: For each i, there will be edges corresponding to $(C_i \cap W, \overline{C_i \cap W})$ and $(C_i \cap \overline{W}, \overline{C_i \cap \overline{W}})$

Local Improvement

Example with $(uv, C_1, C_2, C_3) = (uv, \{a, b, g\}, \{c, e, f\}, \{d, h\})$



- Let $(W, \overline{W}) = (\{a, b, c, d\}, \{e, f, g, h\})$ be the cut of uv
- Combination of Observation 1 and 2:
  - The widths of edges "near the center" will be $f(C_i)$, $f(C_i \cap W)$, and $f(C_i \cap \overline{W})$ for each $i \in \{1, 2, 3\}$

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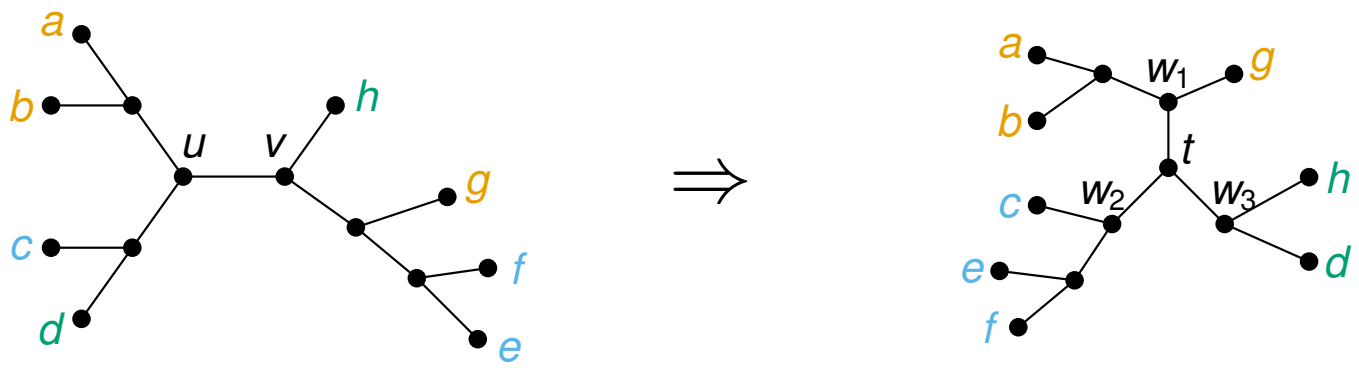

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Theorem

For any set $W \subseteq V$ with $f(W) > 2\text{bw}(f)$ there exists tripartition (C_1, C_2, C_3) of V so that for each i it holds that $f(C_i) < f(W)/2$, $f(C_i \cap W) < f(W)$, and $f(C_i \cap \overline{W}) < f(W)$.

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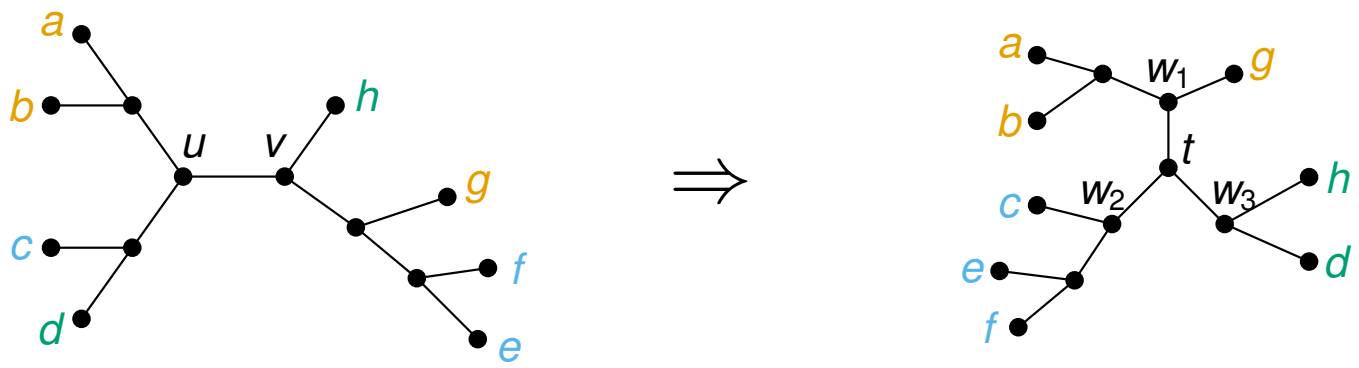

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$\Rightarrow$ If $f(uv) > 2\text{bw}(f)$, there exists refinement with uv that *"locally"* improves T

Global Improvement

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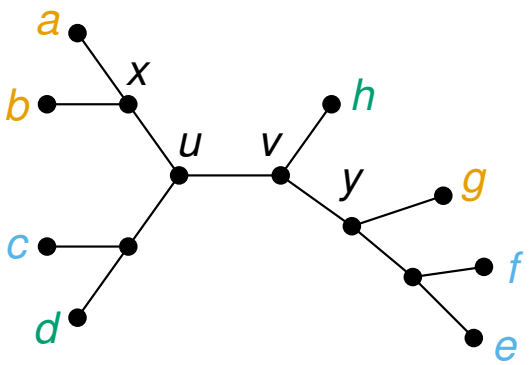
Theorem

If there exists a W-improvement, then there exists a W-improvement (C_1, C_2, C_3) so that refinement with (uv, C_1, C_2, C_3) does not increase width and decreases the number of heavy edges.

Global Improvement: Observation



- Consider T rooted at $r = uv$
- For a node $x \in V(T)$, denote by $T_r[x] \subseteq V$ the leaves in the subtree below x
  - Example: $T_r[x] = \{a, b\}$ and $T_r[y] = \{e, f, g\}$

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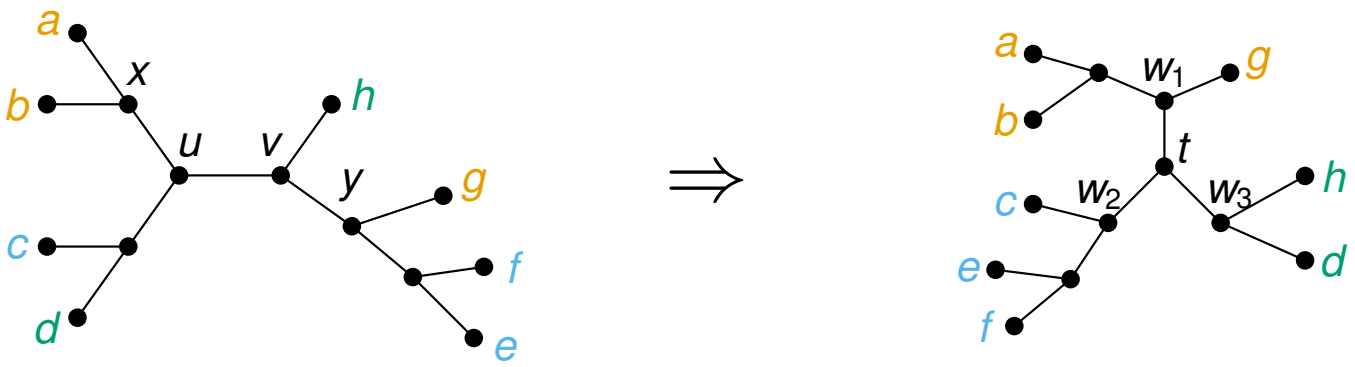

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  - Example: $T_r[x] = \{a, b\}$ and $T_r[y] = \{e, f, g\}$
- Let T' be refinement of T with (uv, C_1, C_2, C_3)
- Observation: Each edge of T' corresponds either to $(C_i, \overline{C_i})$ or to $(T_r[x] \cap C_i, \overline{T_r[x] \cap C_i})$ for some $x \in V(T)$

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- For the edge uv, none of the new edges corresponding to uv has width $f(uv)$
  $\Rightarrow$ Strict improvement

Algorithmic Framework

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- General algorithm for improving a branch decomposition:

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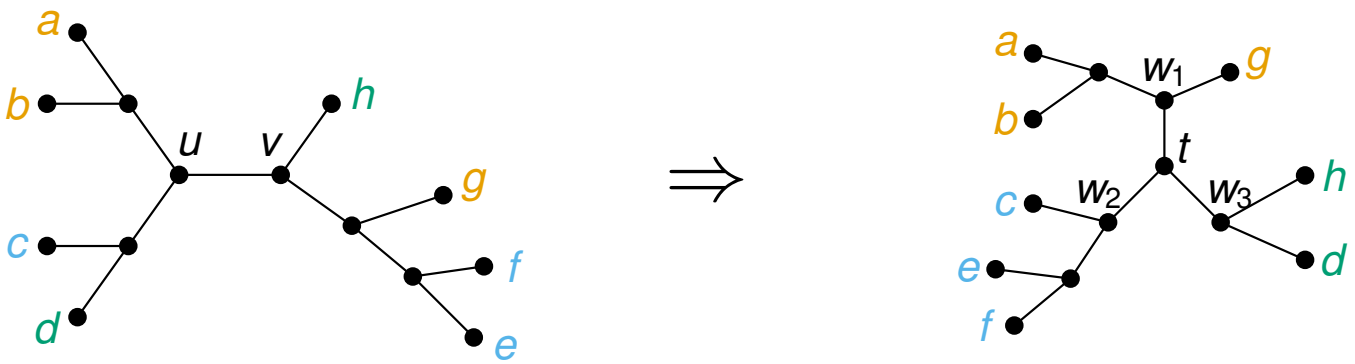
- For graph branchwidth $t(k) = 2^{\mathcal{O}(k)}$

Amortization technique

Example with $(uv, C_1, C_2, C_3) = (uv, \{a, b, g\}, \{c, e, f\}, \{d, h\})$



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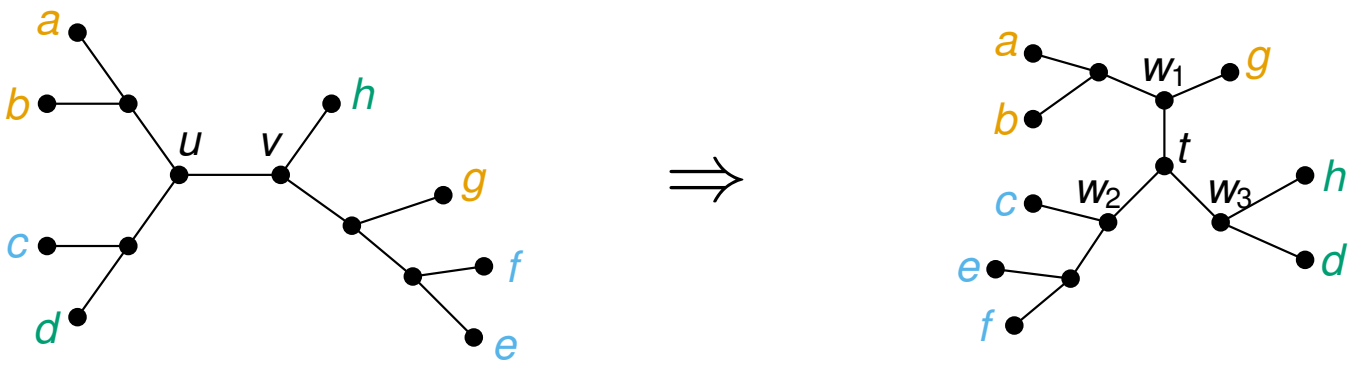

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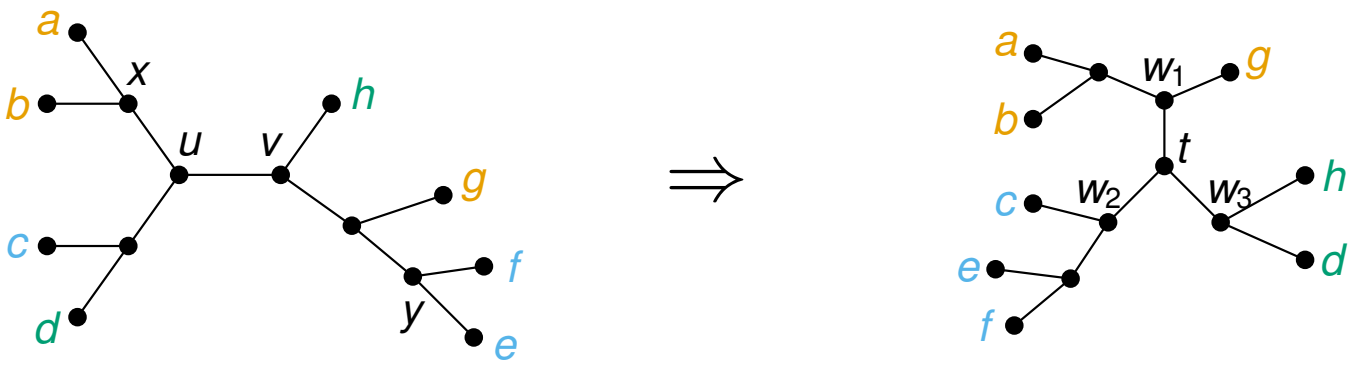

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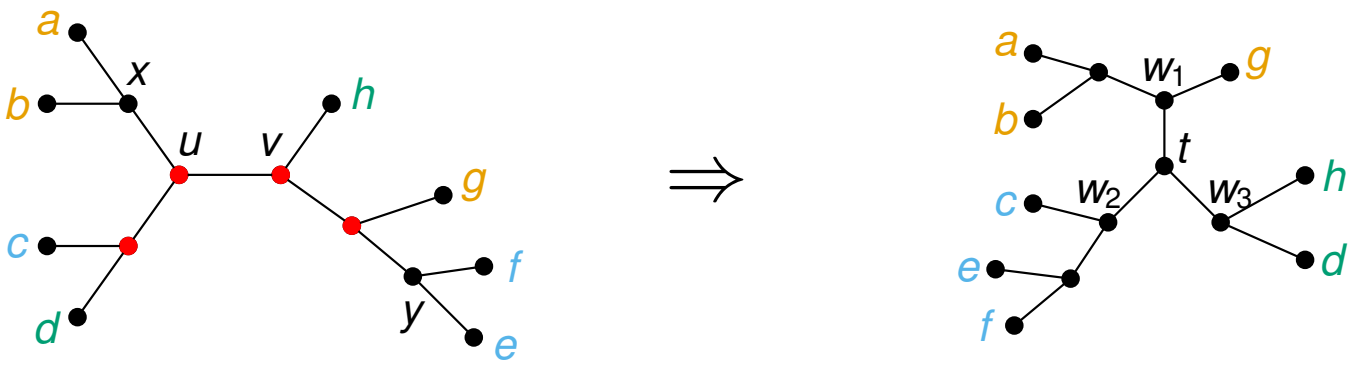

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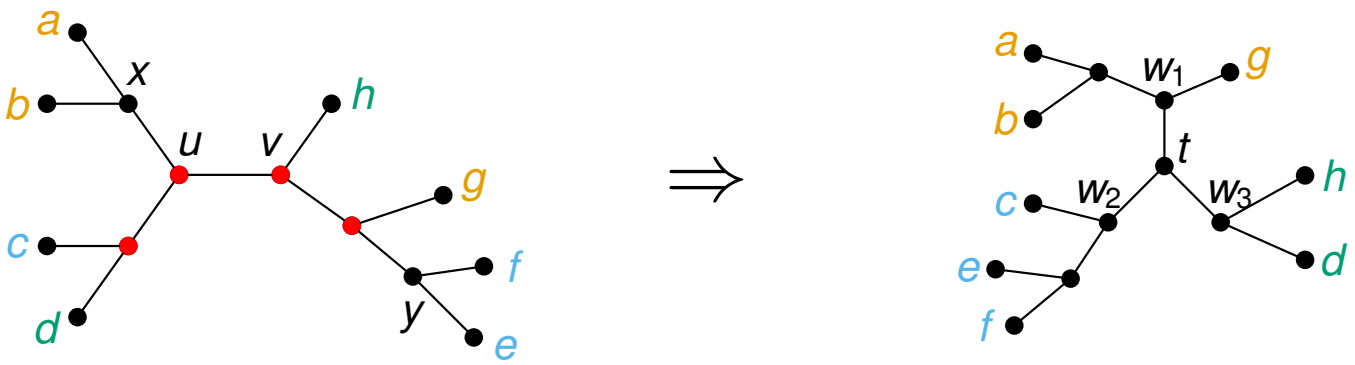

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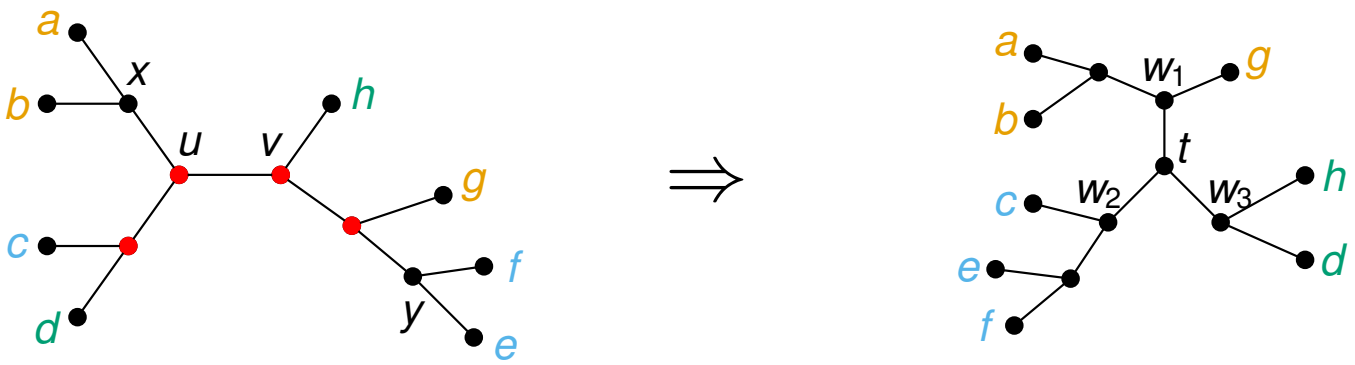

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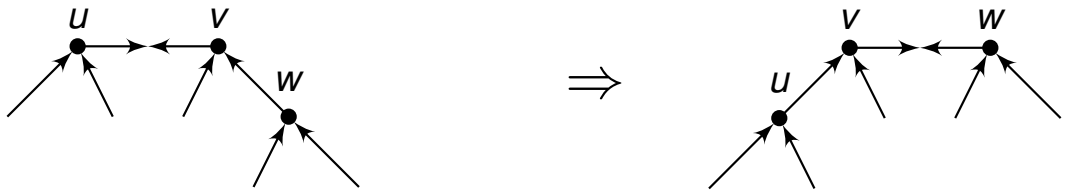

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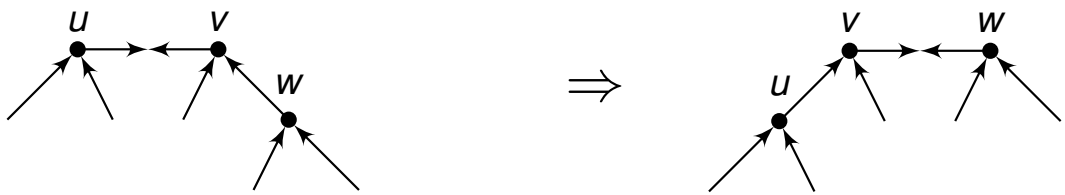

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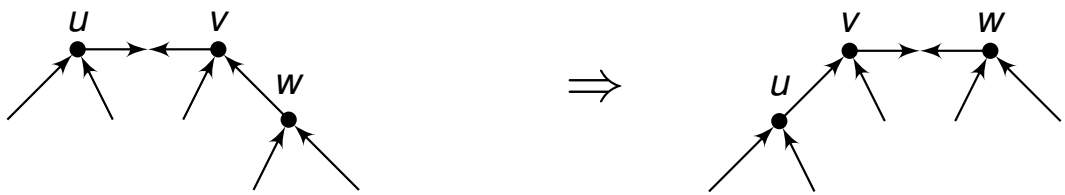

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Thank you!