Dynamic Treewidth in Logarithmic Time

Tuukka Korhonen

IBS DIMAG seminar

9 December 2025

Dynamic graph algorithms

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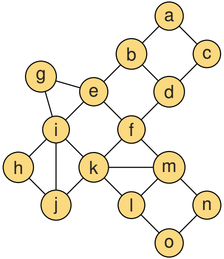
4. [Henzinger&King'99]: $\mathcal{O}(\log^3 n)$ amortized time

Treewidth



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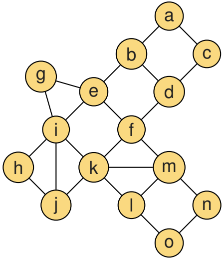

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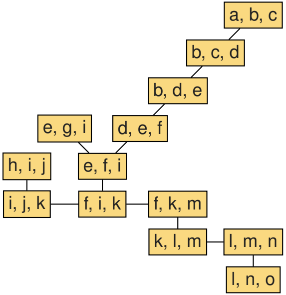

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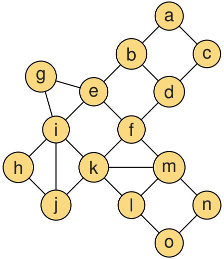

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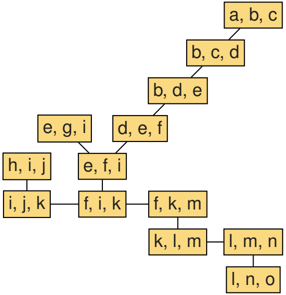

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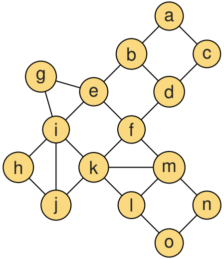

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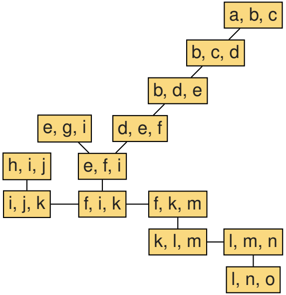

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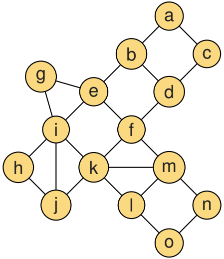

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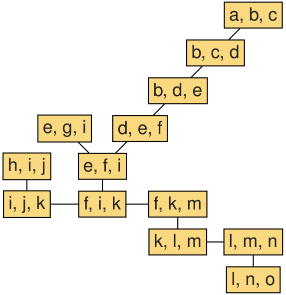

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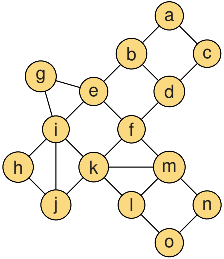

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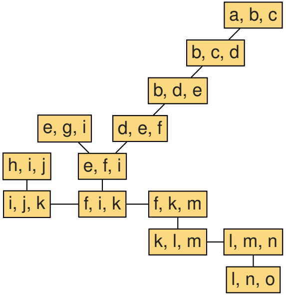

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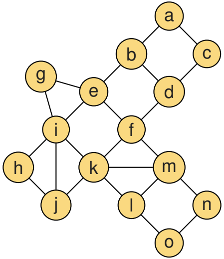

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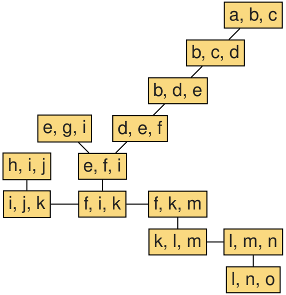

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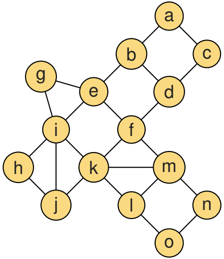

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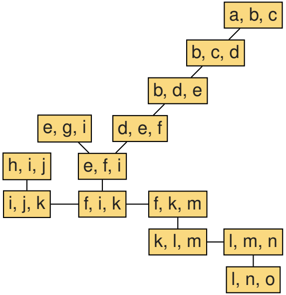

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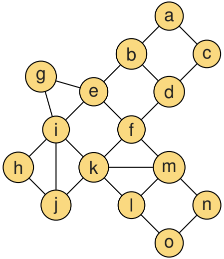

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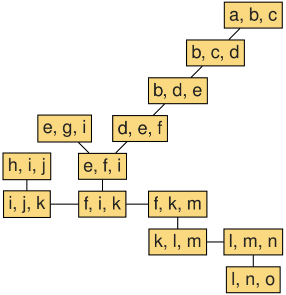

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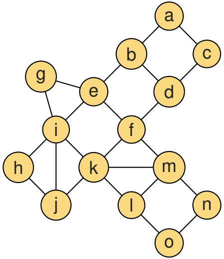

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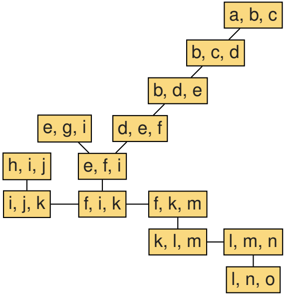

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[Robertson & Seymour'84, Arnborg & Proskurowski'89, Bertele & Brioschi'72, Halin'76]

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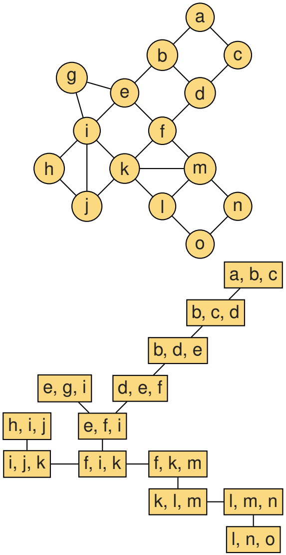

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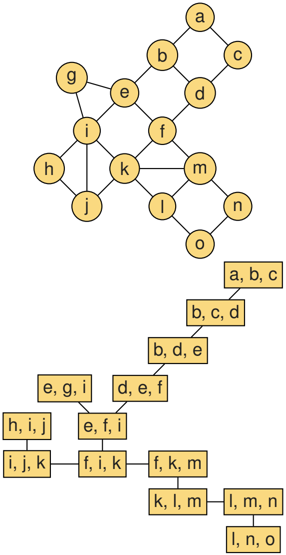

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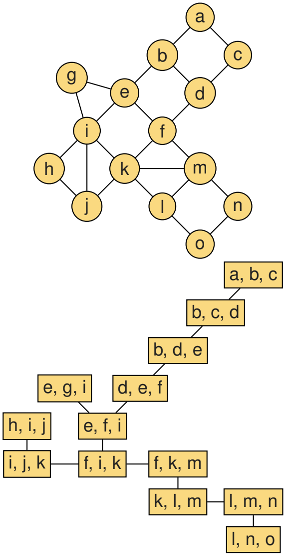

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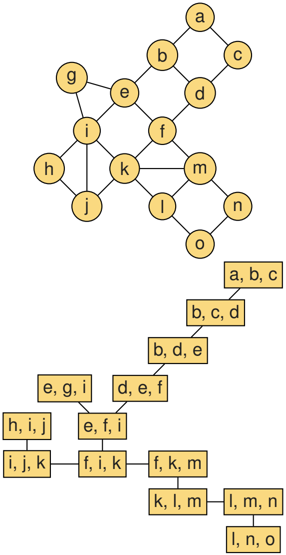

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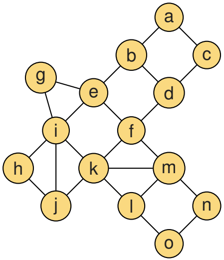
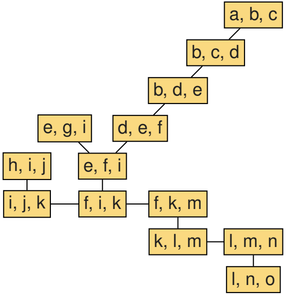
  - $2^{\mathcal{O}(k^3)} n$ time algorithm to compute an optimum-width tree decomposition [Bodlaender '96]
  - $2^{\mathcal{O}(k)} n$ time for 2-approximation [K. '21]
  - $n^{\mathcal{O}(1)}$ time for $\mathcal{O}(\sqrt{\log k})$-approximation [Feige,Hajiaghayi,Lee'08]





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Theorem (This work)

$2^{\mathcal{O}(k)} \log n$ amortized update time 9-approximate tree decomposition.

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There is data structure that

- is initialized with integer k and an edgeless n-vertex graph G
- supports edge insertions/deletions in amortized time $2^{\mathcal{O}(k)} \log n$ under the promise that $\text{tw}(G) \leq k$
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$\Rightarrow$ Dynamic Courcelle's theorem in $f(k) \cdot \log n$ amortized update time

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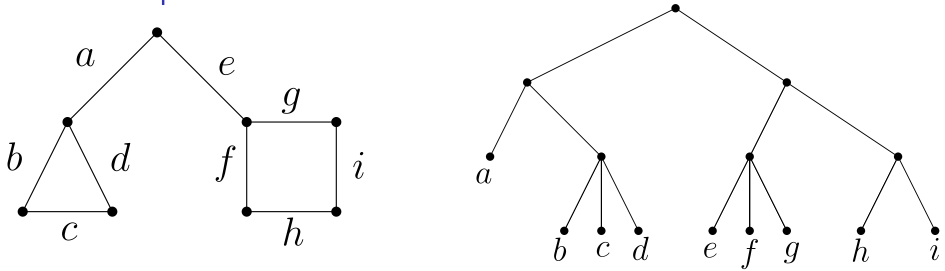
- Dynamic rankwidth $\Rightarrow$ rankwidth in $f(k) \cdot n^{1+o(1)} + \mathcal{O}(m)$ time [K., Sokolowski, '24]
- Dynamic kernelization with $\mathcal{O}(\log n)$ amortized update time, e.g., for planar dominating set [Bertram, Haun, Jensen, K., '25]

The algorithm

Maintained decomposition



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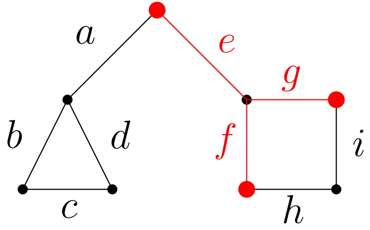

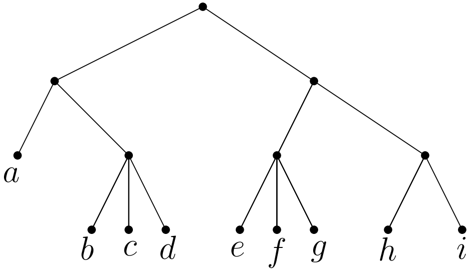

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- Boundary $\partial(F)$ of a set of edges $F \subseteq E$: The vertices incident to edges from both F and $E \setminus F$.

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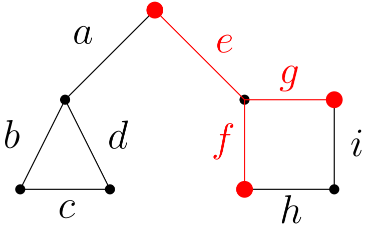

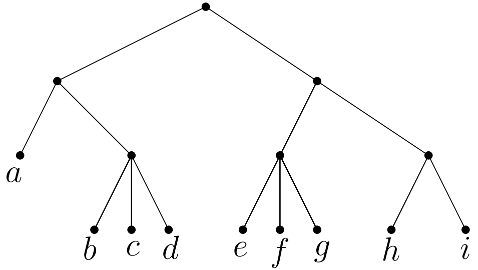

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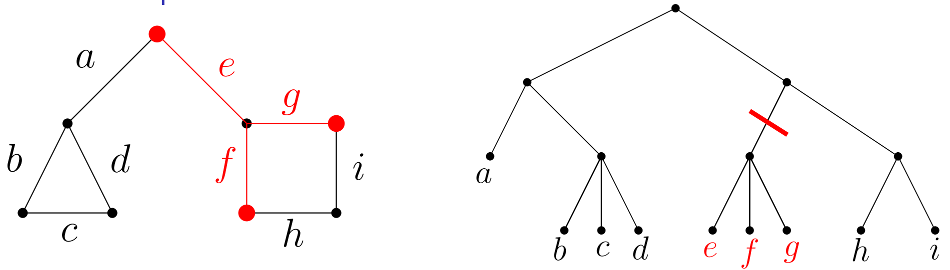

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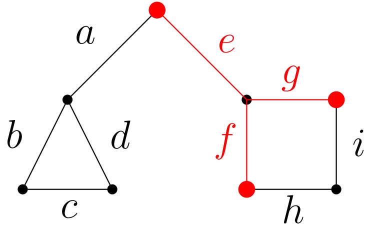

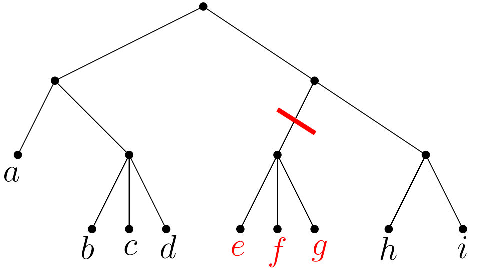

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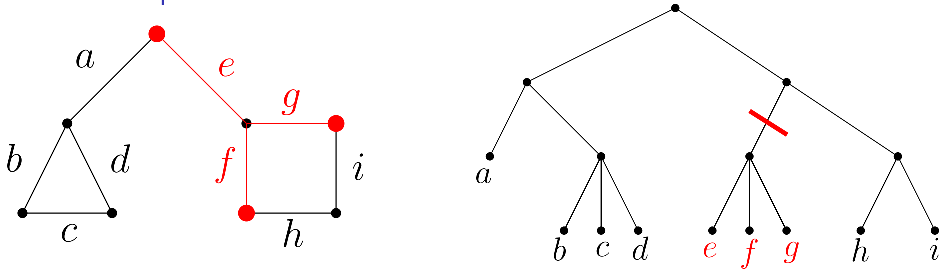

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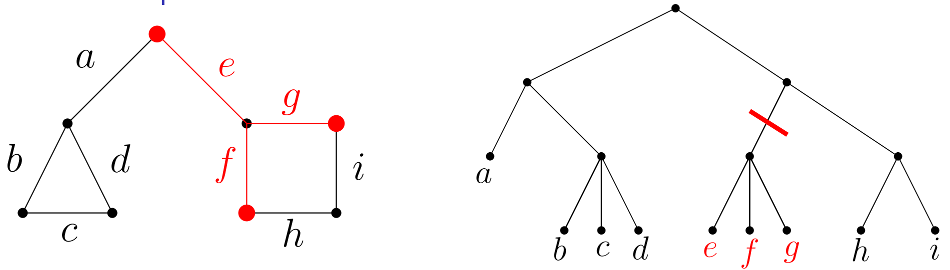

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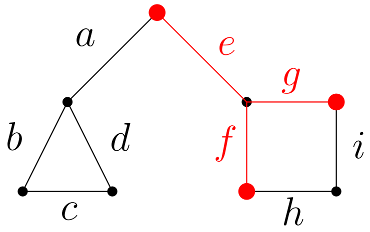

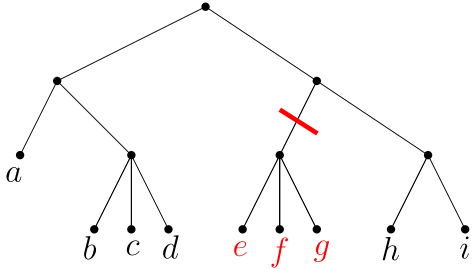

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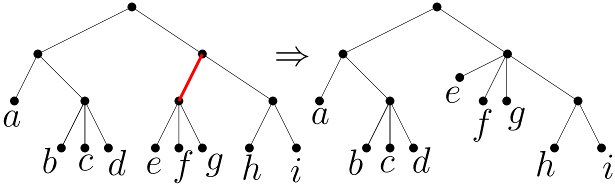

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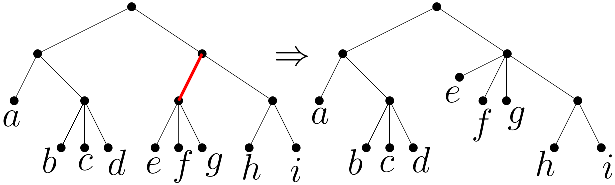

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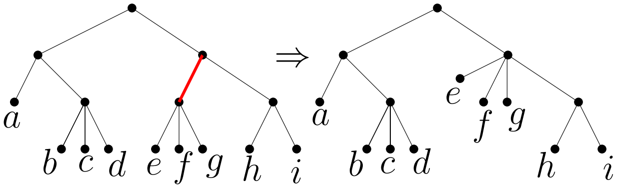

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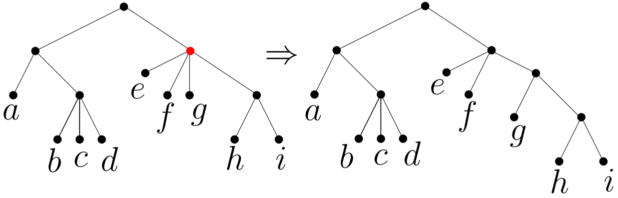

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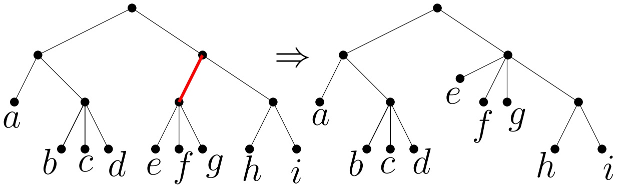

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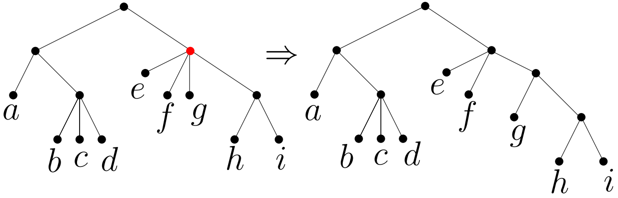

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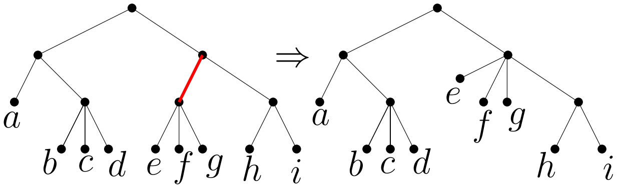

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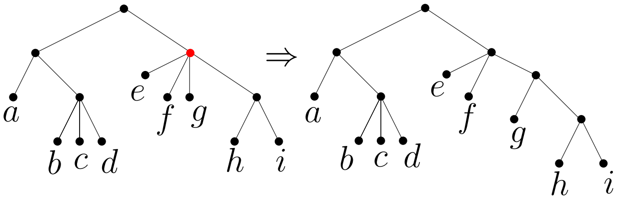

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