Computing Tree Decompositions with Small Independence Number

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joint work with
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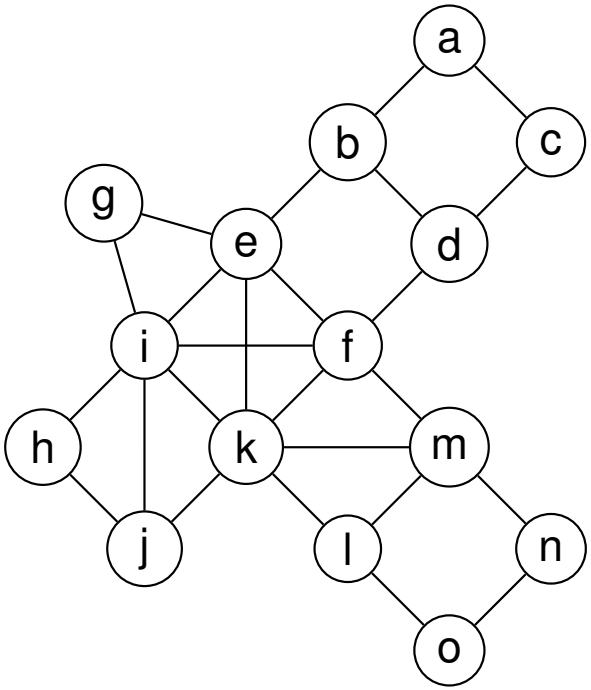
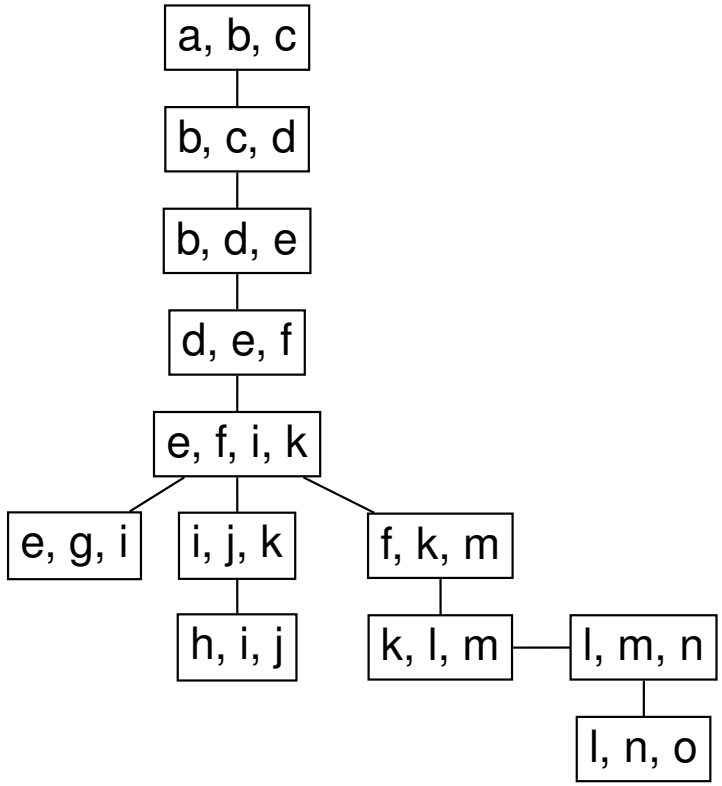
Tree Decompositions

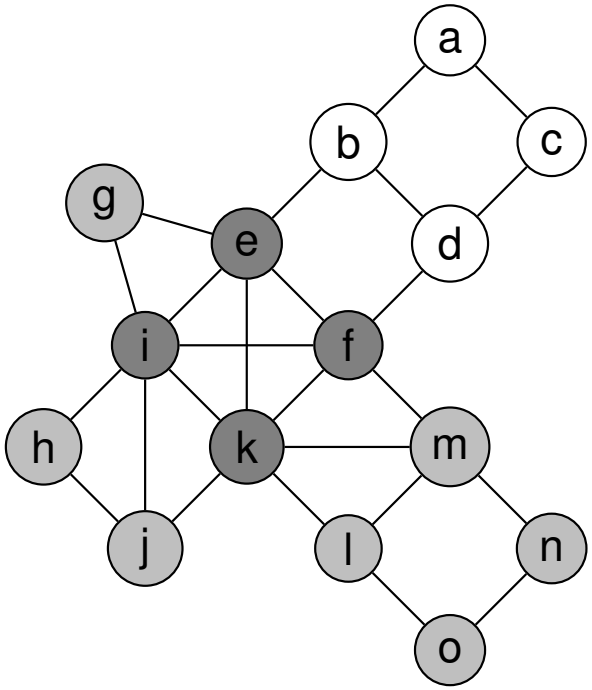


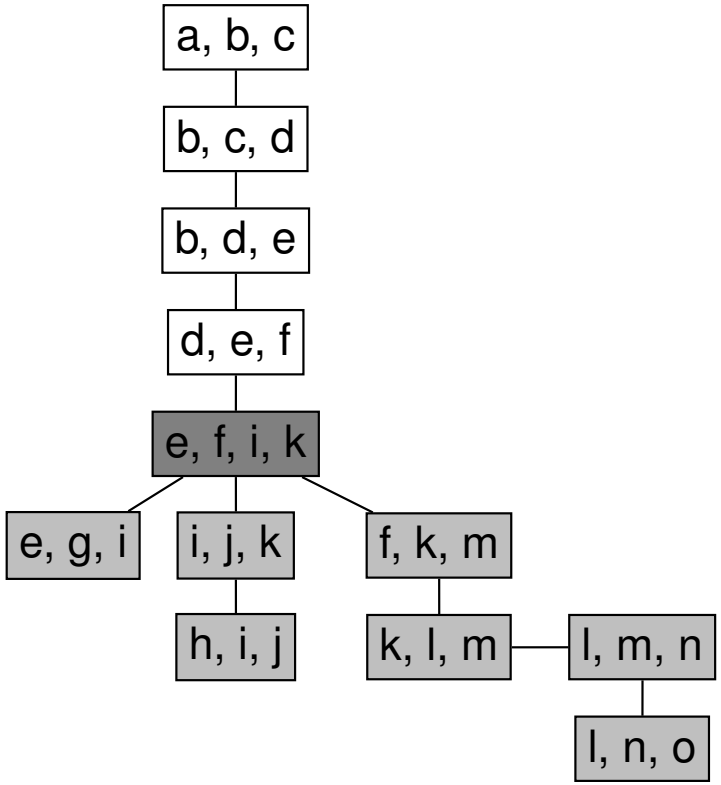
Graph G



A tree decomposition of G

Dynamic programming for maximum independent set





For every node t and subset $S \subseteq B_t$

$\mathrm{dp}[t][S] =$ maximum independent set I below t with $I \cap B_t = S$

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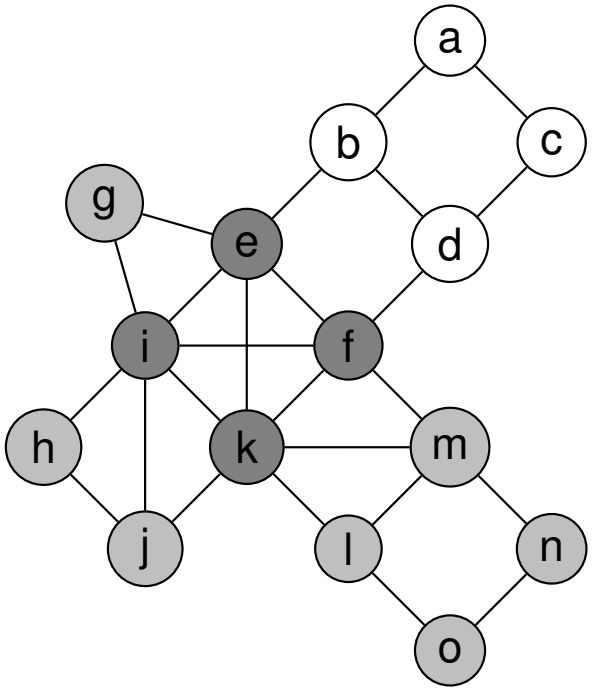

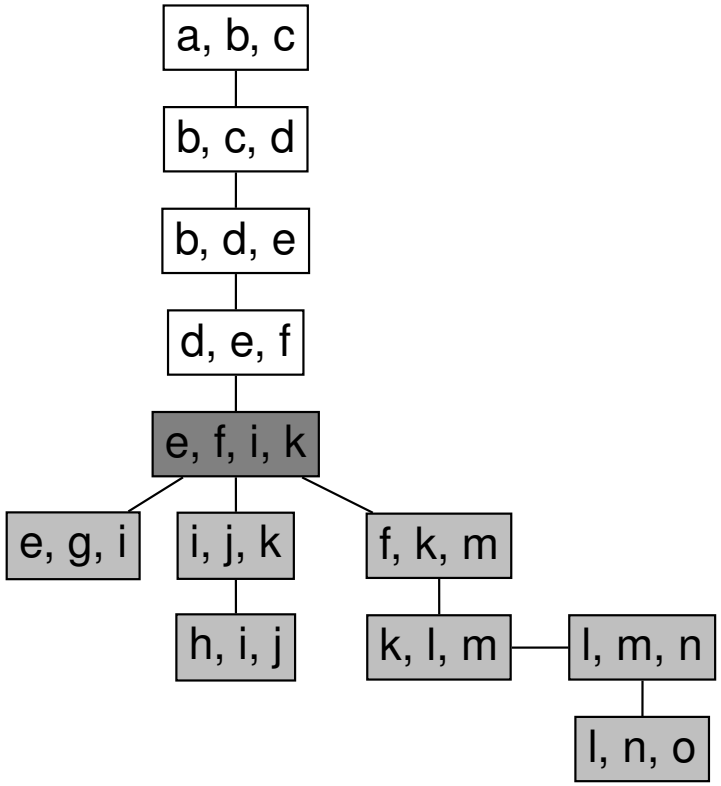

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$2^{|B_t|}$ states per node

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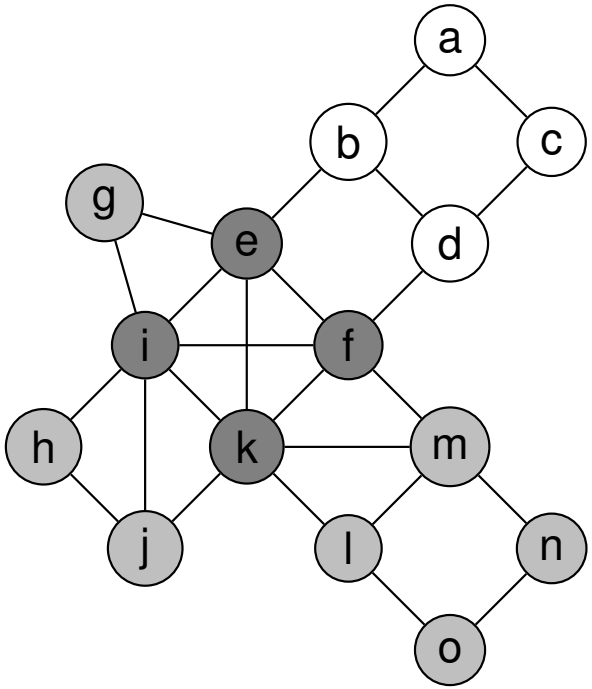

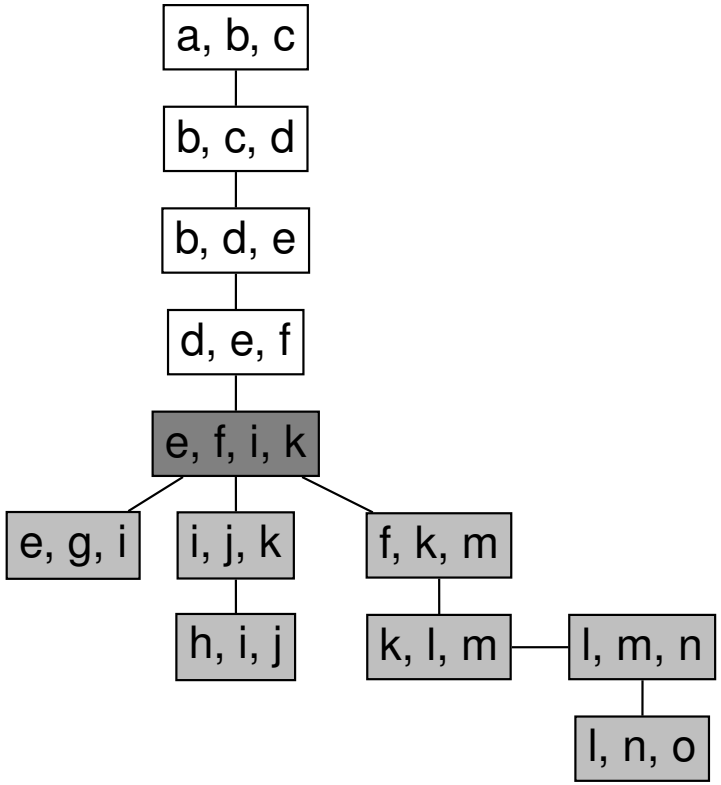

For every node t and **independent** subset $S \subseteq B_t$

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$\#IS(B_t)$ states per node

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All applications need the decomposition as an input!

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Theorem

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  - (For $k = 1$ linear time, $k = 2, 3$ remain open)

The Algorithm

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      3. Linear programming

Balanced separators

Input: Graph G, integer k, and a vertex set X with $\alpha(X) = 9k$

Task: Find a separation (C_1, S, C_2) with $\alpha(S) \leq 2k$, $\alpha(X \cap C_1) \leq 7k$, and $\alpha(X \cap C_2) \leq 7k$ or conclude tree-$\alpha(G) > k$

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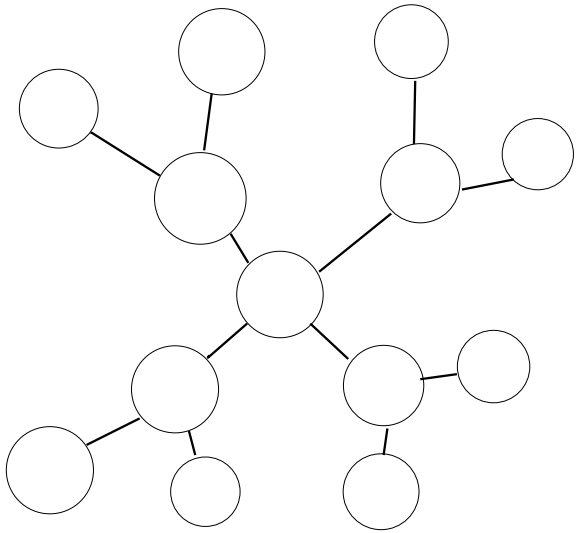

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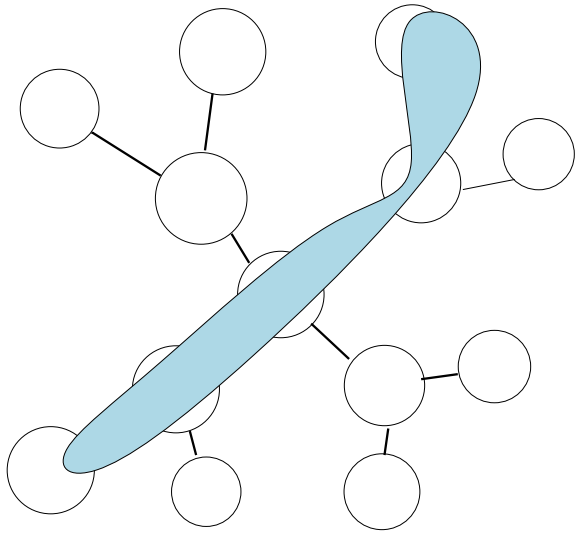

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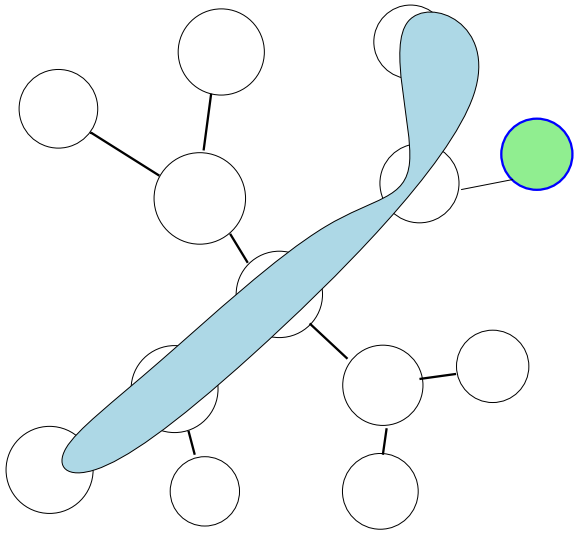

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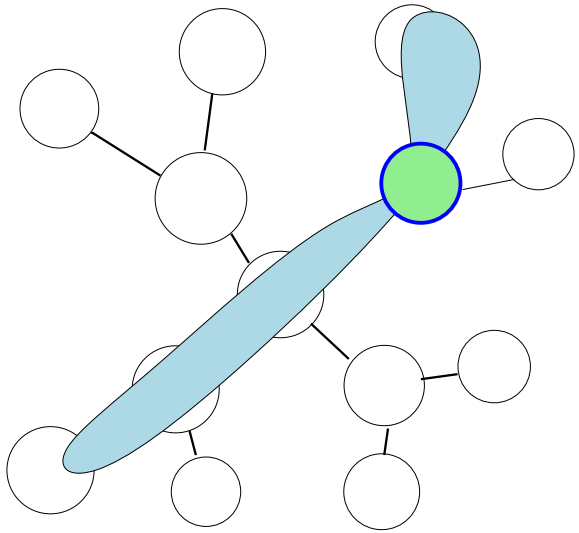

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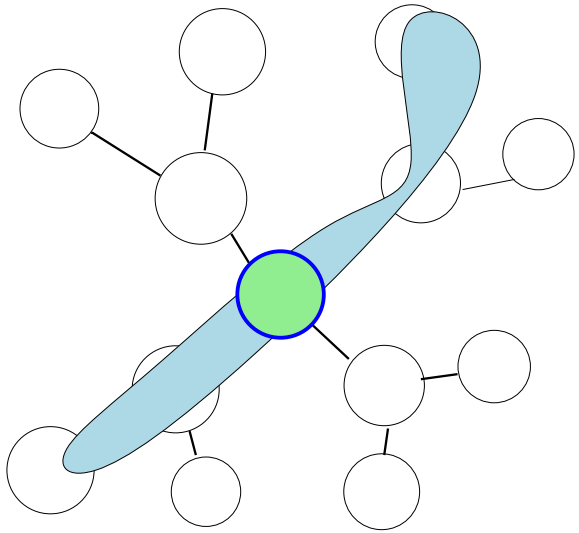

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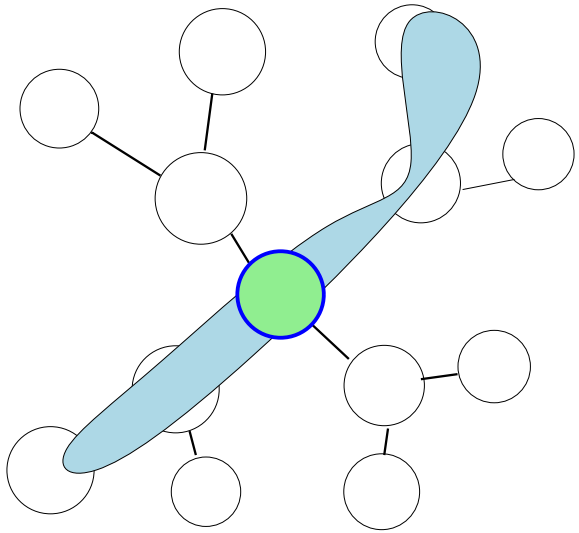

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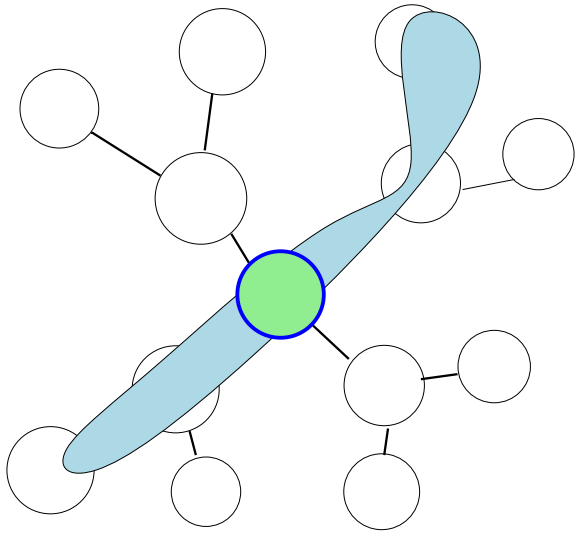

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Algorithm: Guess independent set $I_1 \subseteq X \cap C_1$ with $|I_1| = 2k$ and $I_2 \subseteq X \cap C_2$ with $|I_2| = 2k$, and then find an $I_1 - I_2$ separator S with $\alpha(S) \leq 2k$

2-Approximation Algorithm for separators

Input: Graph G, integer k, and two sets of vertices V_1, V_2

Task: Find an (V_1, V_2)-separator S with $\alpha(S) \leq 2k$, or conclude that no (V_1, V_2)-separators with $\alpha(S) \leq k$ exist

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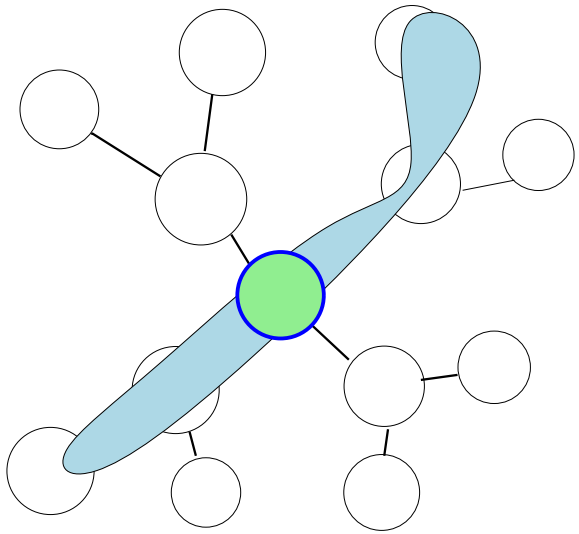

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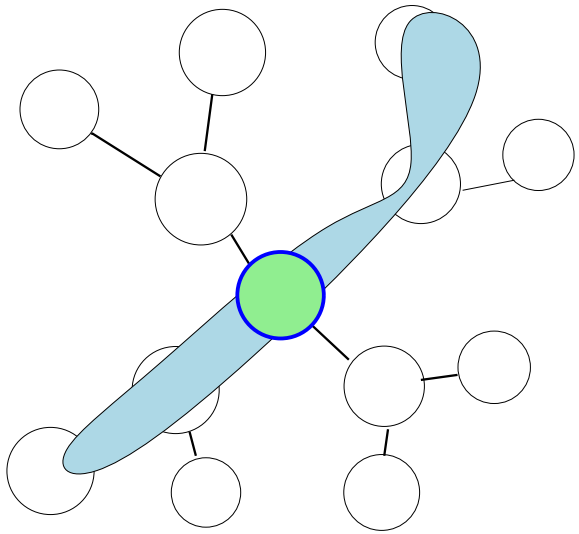

$\Rightarrow$ R can be guessed by guessing $\mathcal{O}(k)$ bags of TD

Branching

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1. v goes to partial solution S_0

Branching

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Branches (2) and (3) decrease $\alpha(R \setminus N(V_1 \cup V_2))$.

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$\Rightarrow$ Branching tree of size $n^{2\alpha(R)}$

Linear Programming

Input: Graph G, integer k, three disjoint sets of vertices V_1, V_2, R with $R = N(V_1 \cup V_2)$

Task: Find an (V_1, V_2)-separator $S \subseteq R$ with $\alpha(S) \leq 2k$, or conclude that no such separators with $\alpha(S) \leq k$ exist

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Lemma

Rounding a fractional solution gives a solution with independence number at most $2k$

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- Testing tree-$\alpha(G) \leq k$ is NP-hard for every $k \geq 4$
- Open problem: Complexity of testing tree-$\alpha(G) \leq k$ for $k = 2, 3$?

Thank you!