A Single-Exponential Time 2-Approximation Algorithm for Treewidth

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In this talk

Theorem. There is an algorithm with:

Input: n-vertex graph G and an integer k.
Output: Tree decomposition of G of width at most $2k+1$, or $\mathrm{tw}(G) > k$.
Running time: $2^{O(k)}n$.

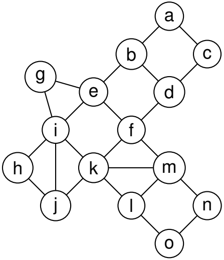
Plan

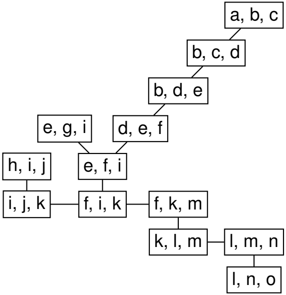
1. Preliminaries and motivation
2. Background and outline of the algorithm
3. Main concepts of the algorithm
4. Implementation in $2^{O(k)} n$ time

Tree Decompositions

A tree decomposition of a graph G is a tree T whose each node $i \in V(T)$ is associated with a bag $B_i \subseteq V(G)$, and satisfies

1. $V(G) = \bigcup_{i \in V(T)} B_i$,
2. for each $\{u, v\} \in E(G)$ there is $i \in V(T)$ with $\{u, v\} \subseteq B_i$, and
3. for each $v \in V(G)$, the induced subgraph $T[\{i \mid v \in B_i\}]$ of T is connected.





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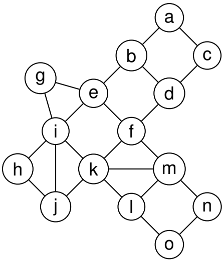
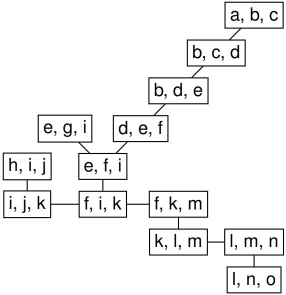
The width of a tree decomposition T is $\max_{i \in V(T)} |B_i| - 1$, and the treewidth of G is the minimum width of a tree decomposition of G.





Applications of treewidth

Given a graph with a tree decomposition of width k, we can solve [Bod88, BCKN15]:

- Maximum independent set in time $2^k k^{O(1)} n$
- Minimum dominating set in time $3^k k^{O(1)} n$
- Steiner tree in time $2^{O(k)} n$
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The approximation ratio c matters:
For $c = 5$, the DP for dominating set takes $3^{5k} k^{O(1)} n = 243^k k^{O(1)} n$ time, while for $c = 2$ it takes $3^{2k} k^{O(1)} n = 9^k k^{O(1)} n$ time.

FPT Approximation Algorithms for treewidth

FPT algorithms for computing a tree decomposition of width at most $\alpha(k)$ in time $f(k)g(n)$, where k is the treewidth of the input graph.

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[BDD+16]	$5k+4$	$2^{O(k)}$	n	$f(k) = \Omega(2^{40k})$
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Part 2: Background and outline of the algorithm

Building Blocks

Bodlaender's compression technique [Bod96]:

Suppose there is a c-approximation algorithm for treewidth with running time $f(k)n$ that requires a tree decomposition of width $2ck$ as an input.

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Then there is a c-approximation algorithm for treewidth with running time $k^{O(1)}f(k)n$ without the requirement.

The new algorithm:

Input: n-vertex graph G and a tree decomposition of G of width w.

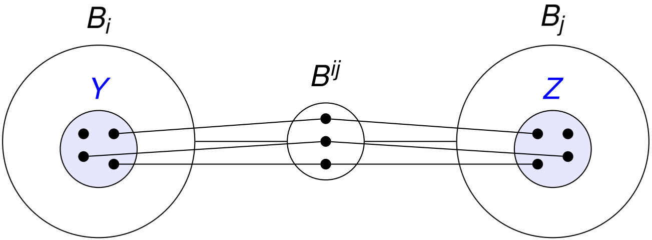
Output: A tree decomposition of G of width at most $w-1$, or $w \leq 2 \cdot tw(G)+1$.

Running time: $2^{O(w)}n$.

Background: Lean Tree Decompositions

Definition (Lean tree decomposition)

A tree decomposition T is *lean* if for any two bags B_i, B_j and vertex sets $Y \subseteq B_i$ and $Z \subseteq B_j$, the number of disjoint paths between Y and Z is $\min(|Y|, |Z|, |B^{ij}|)$, where B^{ij} is a minimum size bag in the path between i and j in T.



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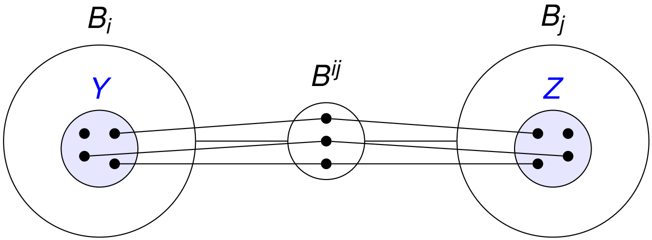
Every graph of treewidth k has a lean tree decomposition of width k.



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We are not interested in the theorem, but on the proof idea of [BD02]:

- Let T be a tree decomposition that is not lean
- $\Rightarrow$ we can apply an improvement step to T
- This does not increase the width of T, and decreases an invariant on T.

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- Let T be a tree decomposition that is not lean
- $\Rightarrow$ we can apply an improvement step to T
- This does not increase the width of T, and decreases an invariant on T.
- Given the "lean witness" (Y, Z), the improvement can be implemented in polynomial time.

The idea: Adaptation of Bellenbaum–Diestel

Observation

If there is a bag B_i of size $|B_i| \geq 3 \cdot tw(G) + 3$, then there is a lean witness (Y, Z) with $Y \subseteq B_i$, $Z \subseteq B_i$.

$\Rightarrow$ 3-approximation algorithm with time complexity $2^{O(k)} n^{O(1)}$ follows quite directly from Bellenbaum–Diestel.

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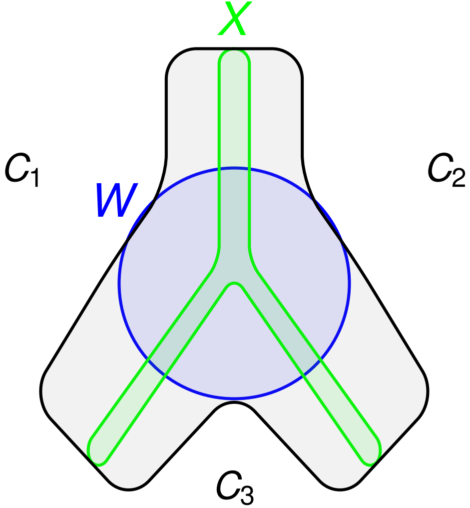
- To achieve approximation ratio 2 instead of 3, instead of lean witnesses we use 3-way separations of bags
- To achieve time complexity $2^{O(k)} n$, we
  - use a "local" version of the improvement
  - refine the invariant analysis
  - use dynamic programming on T to find the separations

Part 3: Main concepts of the algorithm

Splittable Bags

Definition (Splittable bag)

A bag $W \subseteq V(G)$ is *splittable* if $V(G)$ can be partitioned into (C_1, C_2, C_3, X) with no edges between C_i and C_j for $i \neq j$ and $|(W \cap C_i) \cup X| < |W|$ for all $i \in \{1, 2, 3\}$.



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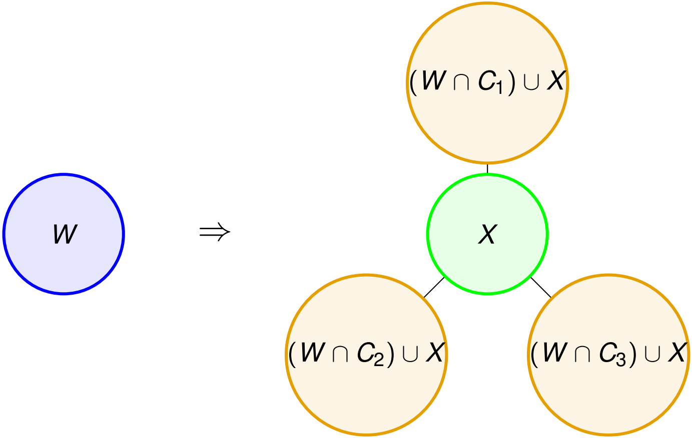
Intuition: Now the following local construction improves the tree decomposition



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Lemma

Any bag W of size $|W| \geq 2 \cdot tw(G) + 3$ is splittable.

Proof:
Let X be a $\frac{1}{2}$-balanced separator of W of size $|X| \leq tw(G) + 1$.

Splitting a Tree Decomposition T

- Let W be the root bag of T and let (C_1, C_2, C_3, X) be a split of W.

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- For each $i \in \{1, 2, 3\}$, obtain a tree decomposition T^i of $G[C_i \cup X]$ by setting $B^i = B \cap (C_i \cup X)$ for each bag B of T.

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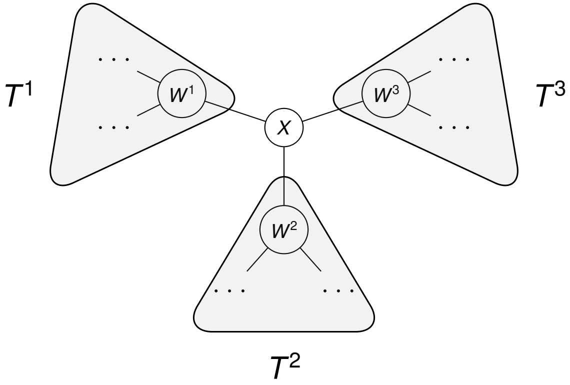
- Now, the following is (almost) a tree decomposition of G:



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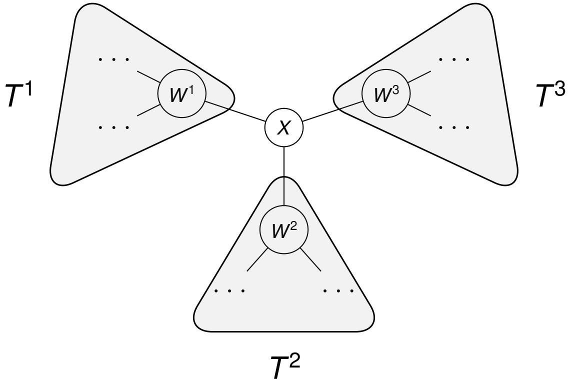
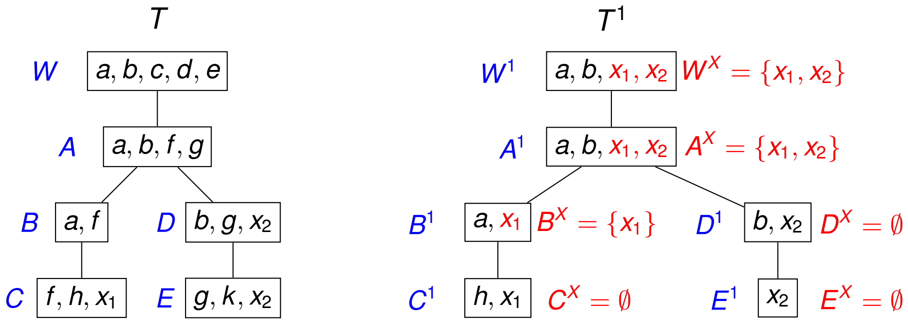


Except that vertices $x_j \in X$ may violate the connected subtree condition

Fixing a tree decomposition

- We fix the connectedness condition by inserting vertices $x_j \in X$ to bags
- Let $B^i = (B \cap (C_i \cup X)) \cup B^X$
- Example: Consider a split $(\{a, b, h\}, \{c, d, f\}, \{e, g, k\}, \{x_1, x_2\})$ of W:



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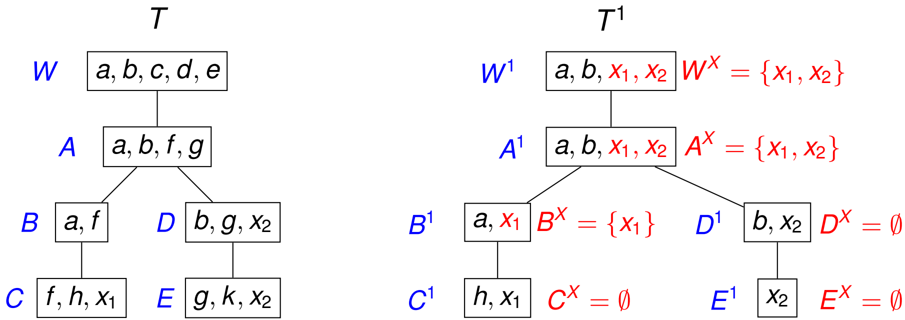
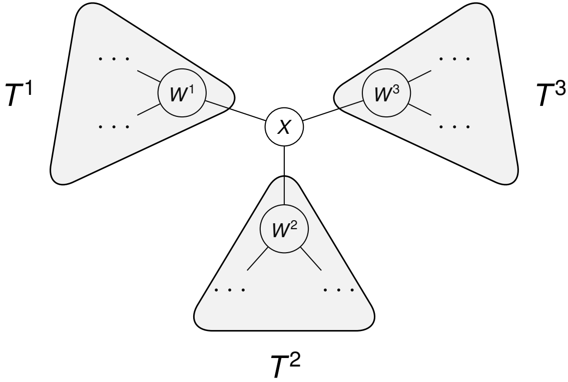


The *home bag* of x_1 is C and the home bag of x_2 is D.

Splitting a Tree Decomposition

Now, with $B^i = (B \cap (C_i \cup X)) \cup B^X$, the following is a tree decomposition of G:



- $|W^i| < |W|$ (and $|X| < |W|$) by the definition of a split.
- It remains to bound $|B^i|$ for the other bags B.

Minimum Splits

We need some extra properties from the split that we use

Definition (Minimum Split)

A split (C_1, C_2, C_3, X) of a bag W is a *minimum split*, if it among all splits of W it

1. primarily minimizes $|X|$, and
2. secondarily minimizes ...

Bounding $|B^i|$

Lemma

Let (C_1, C_2, C_3, X) be a minimum split of W. For all bags B and distinct i, j it holds that $|B^X| \leq |B \cap (C_i \cup C_j)|$.

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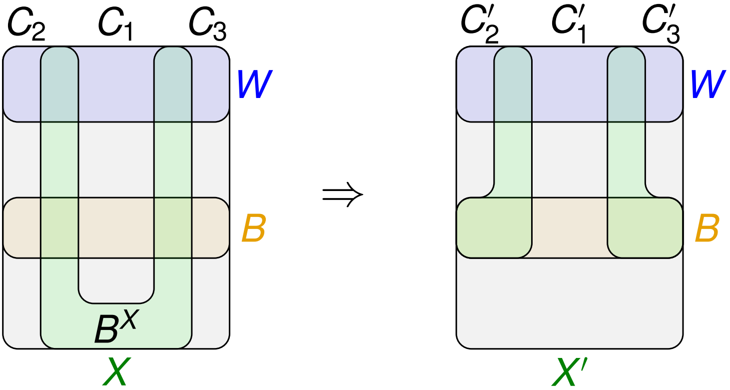
Proof: Suppose $|B^X| > |B \cap (C_2 \cup C_3)|$. Take $X' = (X \setminus B^X) \cup (B \cap (C_2 \cup C_3))$. We have $|X'| < |X|$, so a split (C_1', C_2', C_3', X') contradicts the minimality.



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Where $d_T(x, W)$ is the distance in T from the home bag $H(x)$ of x to W.

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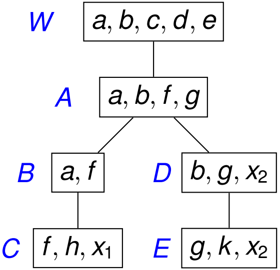
Where $d_T(x, W)$ is the distance in T from the home bag $H(x)$ of x to W.

Example: Here $H(x_1) = C$ and thus $d_T(x_1, W) = 3$, and $H(x_2) = D$ and thus $d_T(x_2, W) = 2$. Therefore $\sum_{x_j \in X} d_T(x_j, W) = 5$.



Bounding $|B^i|$: Stronger Bound

Lemma

Let (C_1, C_2, C_3, X) be a minimum split of W. For all bags B with non-empty B^X and distinct i, j it holds that $|B^X| < |B \cap (C_i \cup C_j)|$

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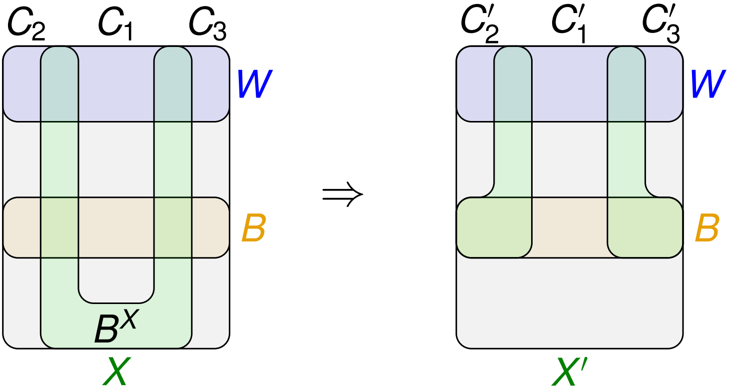
The proof is exactly same as before, with the observation that $d_T(y, W) < d_T(x, W)$ for all $y \in B$ and $x \in B^X$.



Bounding $|B^i|$: Summary

By using minimum splits we have that:

1. $|B^i| \leq |B|$ for all bags B, and
2. $|B^i| < |B|$ if B^X is non-empty, implying
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$\Rightarrow$Splitting a largest bag W of T does not increase the width of T, and decreases the number of bags of size $|W|$.

$\Rightarrow$If we implement splitting in time $2^{O(w)}n$, we obtain a $2^{O(k)}n^2$ time 2-approximation algorithm for treewidth.

Part 4: Implementation in $2^{O(k)}n$ time

Issue 1: Number of bag edits

Issue: By implementing the splits as described before, solely the bag editing will take $\Omega(n^2)$ time over the course of the algorithm.

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Solution: Modified bag editing process:

- We define *editable* bag
- Each splitting operation will take $w^{O(1)}t$ time, where t is the number of editable bags of the split.
- $t \leq \phi(T) - \phi(T')$, where T is the old tree decomposition, T' is the new tree decomposition, and ϕ is a potential function.
- In particular, $\phi(T) = \sum_{i \in V(T)} 7^{|B_i|}$.

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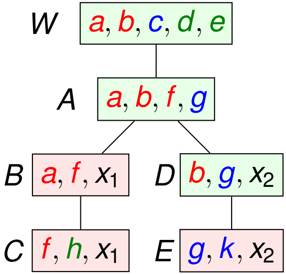
$\Rightarrow$ Time taken by bag editing will be bounded by $\phi(T)w^{O(1)} = 2^{O(w)}n$.

Editable bags

Definition

Let (C_1, C_2, C_3, X) be a split of a root bag W. A bag B is *editable* if it intersects at least two of C_1, C_2, C_3, and its parent is editable.

Example: Consider a split $(\{a, b, f\}, \{c, g, k\}, \{d, e, h\}, \{x_1, x_2\})$.



Here bags W, A, and D are editable.

Re-Assigning Vertices

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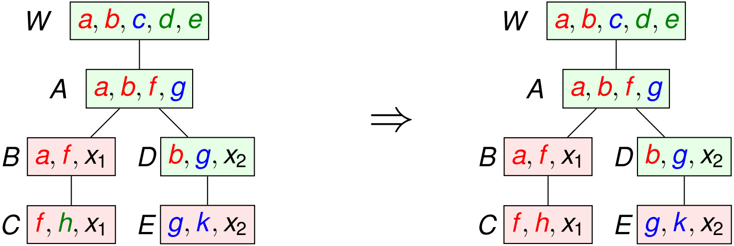
Consider a non-editable bag B whose parent A is editable

$(\{a, b, f\}, \{c, g, k\}, \{d, e, h\}, \{x_1, x_2\})$ $\quad$ $(\{a, b, f, h\}, \{c, g, k\}, \{d, e\}, \{x_1, x_2\})$



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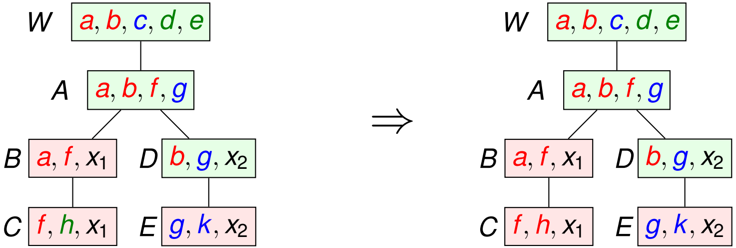

We re-assign h from C_3 to C_1.

With this re-assignment, a bag is editable if and only if it intersects at least two of C_1, C_2, C_3. Also, the editable bags form a connected subtree containing W.

Editing Editable Bags

Observation: For non-editable bags in minimum splits, $B^X = \emptyset$.

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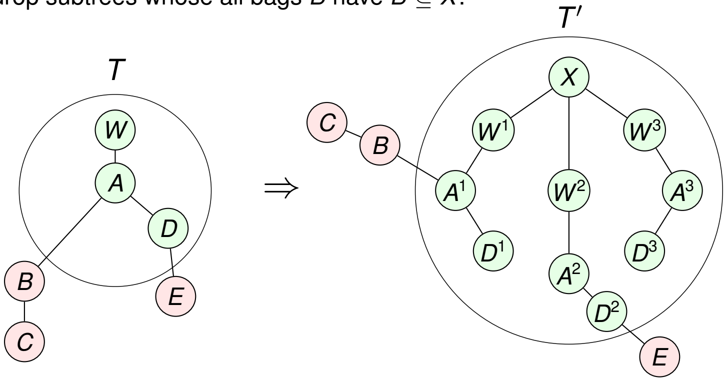
We can drop subtrees whose all bags B have $B \subseteq X$.



Now the splitting can be done by removing the t editable bags, inserting $3t+1$ new bags, and connecting the non-editable subtrees accordingly.

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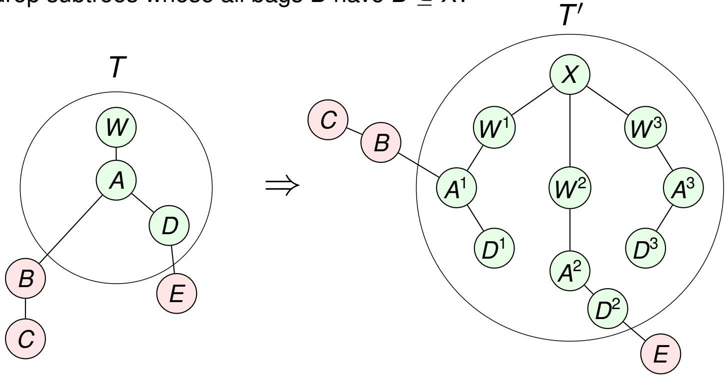

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Detail: To do this in $w^{O(1)}t$ time, we will maintain that T has maximum degree 3 (this increases the number of new bags inserted to $3t + 4$).

Analysis of the Potential Function

Consider the potential function $\phi(T) = \sum_{i \in V(T)} 7^{|B_i|}$.

Because $|B^i| < |B|$ for editable bags, simple computation shows $\phi(T) - \phi(T') \geq t$ where t is the number of editable bags.

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$\Rightarrow$ The algorithm can be implemented so that we edit $2^{O(w)}n$ bags in total over the course of the algorithm.

Issue 2: Finding the splits

Issue: We may need to perform $\Omega(n)$ split operations to decrease the width, so each split should be found in (amortized) $2^{O(w)}$ time.

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Solution: Use dynamic programming on the tree decomposition while editing it

- Using dynamic programming on the tree decomposition, we can compute a minimum split of the root bag in $2^{O(w)}n$ time.
- We can re-root the dynamic programming to an adjacent node in $2^{O(w)}$ time
- DFS-style procedure to with a dynamic programming-based data structure to implement everything in $2^{O(w)}n$ total time.

Data Structure

We maintain a degree-3 width $\leq w$ tree decomposition T rooted at node r.

Operation	Time	Description
Init(T, r)	$2^{O(w)}n$	Given a degree-3 width w tree decomposition T and a root node $r \in V(T)$, initialize the data structure.
Move(s)	$2^{O(w)}$	Move the root r to a neighboring node s.
Split()	$2^{O(w)}$	If the bag W of r is splittable set the internal state to a minimum split (C_1, C_2, C_3, X) of W and return $\top$. Otherwise return $\perp$.
State()	$w^{O(1)}$	Returns the internal state restricted to the bag W of r, i.e., the partition $(C_1 \cap W, C_2 \cap W, C_3 \cap W, X \cap W)$.
Edit(T_1, T_2, r')	$2^{O(w)}(\lvert T_1\rvert + \lvert T_2\rvert)$	Replaces a subtree T_1 with a given subtree T_2. Assumes that $r \in V(T_1)$, and places new r to r'.

Dynamic Programming

Let i be a node of T and $G[T_i]$ the subgraph of G induced by vertices in the subtree of T rooted at i.

$$dp[i][(C_1 \cap B_i, C_2 \cap B_i, C_3 \cap B_i, X \cap B_i)][h]$$

Stores either

1. The minimum value of $\sum_{x \in X} d_T(H(x), B_i)$ such that there is a partition (C_1, C_2, C_3, X) of $G[T_i]$ with no edges between C_i, C_j when $i \neq j$ and $|X| = h$ or
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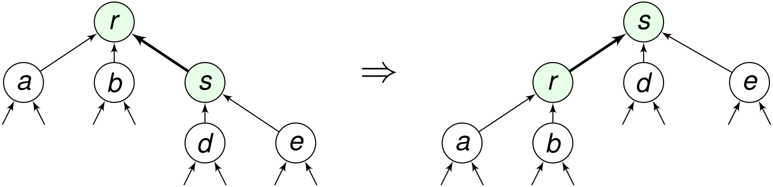

Moving the root from r to s requires updating tables of r and s.

Putting All Together

- DFS over the tree decomposition, every time we see a bag of size $w + 1$ we apply the splitting operation.
- By the potential function, the total number of nodes considered will be $2^{O(w)}n$
- Each move in the tree decomposition takes $2^{O(w)}$ time
- Total time complexity $2^{O(w)} \cdot 2^{O(w)}n = 2^{O(k)}n$.

Conclusion

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- Future ideas: The approximation ratio 2 appears only in the lemma that guarantees every bag of size $\geq 2 \cdot tw(G) + 3$ is splittable. Can we do better analysis about what happens if we just continue splitting bags until no bag is splittable?
- For this purpose, the bag splitting can be generalized to arbitrary number of parts.

The end

Thank you for your attention!

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