Computing Tree Decompositions with Small Independence Number

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based on joint work with
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Vertex Partitioning in Graphs: From Structure to Algorithms

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- Generalizes chordal graphs, treewidth, and various prior generalizations of chordal graphs

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- $\mathcal{O}(n^{|H|\cdot(k+2)})$ time maximum weight H-packing [Dallard, Milanič & Storgel '21]
- $f(k) \cdot n^{\mathcal{O}(k)}$ time algorithms for
  - feedback vertex set
  - longest induced path
  - maximum weight induced subgraph with bounded chromatic number satisfying a CMSO property [Milanič & Rzążewski '22]

Our results on computing tree-independence number

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There is a $2^{\mathcal{O}(k^2)} n^{\mathcal{O}(k)}$ time 8-approximation algorithm for tree-independence number, which also outputs the corresponding tree decomposition.

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- Complexity of deciding tree-$\alpha(G) \leq k$ for $k = 2, 3$?
- Is deciding tree-$\alpha(G) \leq k$ for unbounded k in NP or $\sum_2^P$-hard?

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4. 2-approximation algorithm for separators

Outline of 2-approximation of separators

Input: Graph G, integer k, and two sets of vertices V_1, V_2

Task: Find an (V_1, V_2)-separator S with $\alpha(S) \leq 2k$, or conclude that no (V_1, V_2)-separators with $\alpha(S) \leq k$ exist

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- Proof: Independent set is NP-hard on twice-subdivided graphs, but they have a tree decomposition where every bag induces either independent set or P_4

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- Based on a theorem that a maximal independent set of G can intersect S in at most $n^{\mathcal{O}(\mu(S))}$ different ways

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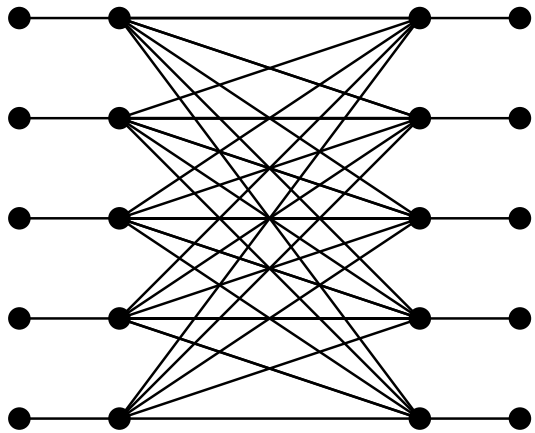

Thank you!